# *Chapter 10*

# DIRECT AND CHAINED INDICES: A REVIEW OF TWO PARADIGMS

Bert M. Balk[1]

## 1. Introduction

A recurrent theme when measuring aggregate price and quantity change between more than two periods is the choice between the computation of direct or chained index numbers. Suppose we consider periods 0, 1, 2, …, $T$ and want to measure change relative to the base period $0$. A direct index number comparing period $t$ ($t = 1,\ldots,T$) to period $0$ results from inserting period $t$ and period 0 data into a bilateral index formula. A chained index number comparing period t to period 0 results from successively inserting period 1 and period 0 data, period 2 and period 1 data, …., and period $t$ and period $t$-1 data into a bilateral index formula and multiplying the outcomes with each other.

A commonly claimed advantage of the method of chaining is the reduction of so-called index number spread. As the *CPI Manual* (2004) states:

> "The main advantage of the chain system is that under normal conditions, chaining will reduce the spread between the Paasche and Laspeyres indices." (par. 15.83)
>
> "Basically, chaining is advisable if the prices and quantities pertaining to adjacent periods are *more similar* than the prices and quantities of more distant periods, since this strategy will lead to a narrowing of the spread between the Paasche and Laspeyres indices at each link." (par. 15.85)

The detailed numerical example discussed in chapter 19 of the *CPI Manual* also reflects this viewpoint, as the following quotations make clear:

> " … if the underlying price and quantity data are subject to reasonably smooth trends over time, then the use of chain indices will narrow considerably the dispersion in the asymmetrically weighted indices." (par. 19.16)

---

[1] Rotterdam School of Management, Erasmus University, and Statistics Netherlands, The Hague. Email: bbalk@rsm.nl. The views expressed in this paper are those of the author and do not necessarily reflect those of Statistics Netherlands. The author thanks Erwin Diewert for comments on a previous version.

Citation: Bert M. Balk (2010),
"Direct and Chained Indices: A Review of Two Paradigms,"
chapter 10, pp. 217-234 in
W.E. Diewert, B.M. Balk, D. Fixler, K.J. Fox and A.O. Nakamura (2010),
PRICE AND PRODUCTIVITY MEASUREMENT: Volume 6 -- Index Number Theory. Trafford Press.
Also available as a free e-publication at www.vancouvervolumes.com and www.indexmeasures.com.

> " … the combined effect of using both the chain principle as well as symmetrically weighted indices is to dramatically reduce the spread between all indices constructed using these two principles." (par. 19.21)

The overall impression one gets is that chained index numbers are somehow closer to the truth than direct index numbers. But is this impression warranted?

The technique of chaining index numbers was introduced by Lehr (1885) and Marshall (1887) primarily as a means to overcome the problems of making comparisons for distant periods when there are many disappearing and newly appearing commodities through time. Statistical agencies were reluctant to officially use chained index numbers. However, during the last two decades this situation has started to change.

The growing acceptance of chained index numbers was not brought about by some convincing theoretic demonstration of the 'verisimilitudiness' of the method of chaining. Instead, under the influence of a small number of researchers, some important agencies in the field of economic measurement changed their ways.

Both the use of chaining and the replacement of the Laspeyres and Paasche indices (which are asymmetrically weighted) by Fisher indices (which are symmetrically weighted) are practices that have met with criticism from some, notably Peter von der Lippe[2]. Certainly it would be helpful to know more about how the approaches compare.

The plan of this paper is as follows. Section 2 summarizes the traditional point of view that is based on the use of direct Laspeyres and Paasche indices. Section 3 summarizes the modern point of view based on the use of chained Fisher indices. In section 4, the two views are compared. The conclusion that emerges is that, mathematically at least, a unification of the two approaches is impossible. A related question that remains to be answered is: What precisely does a chained price or quantity index measure? I search for an answer in section 5 using micro-economic theory, and in section 6 using Divisia index theory. Section 7 concludes.

## 2. The Traditional Point of View

I consider an economic aggregate consisting of a number of transaction categories that I will call 'commodities'. For the time being, I will assume that these commodities do not change through time. Each commodity ($n = 1,\ldots,N$) has an (average) price $p_n^t$ per unit in each period t and a corresponding quantity $q_n^t$ measured using the same units. The superscript t denotes the time period (thought of here as being a year). The (transaction) value of commodity n in period t is then $v_n^t \equiv p_n^t q_n^t$, and the value of the entire aggregate is $V^t \equiv \sum_{n=1}^N p_n^t q_n^t$. It is efficient to use from hereon simple vector notation. Hence, $p^t \cdot q^{t'} \equiv \sum_{n=1}^N p_n^t q_n^{t'}$, where $t$ and $t'$ denote two, not necessarily different, time periods.

---

[2] See Von der Lippe (2000), (2001a), (2001b), Reich (2000), and Rainer (2002). The discussion appears to be by and large limited to the readership of the *Allgemeines Statistisches Archiv*. A recent summary of Von der Lippe's position is provided by Von der Lippe (2007, Chapter 7).

Consider now the development of this aggregate through a number of consecutive periods, say $t$ = 0, 1, 2, …., $T$. The associated sequence of nominal values is given by

(1) $$p^0 \cdot q^0,\ p^1 \cdot q^1,\ p^2 \cdot q^2, \ldots.,\ p^T \cdot q^T .$$

It is clear that the nominal value development is caused by both price and quantity changes. The problem is to disentangle the two components in order to get a picture of the 'real', quantity part of the development.

The traditional solution[3] involves transforming the sequence of nominal values into a sequence of values-at-constant-prices. If one employs the period 0 prices as constant prices, the solution becomes that of computation of the sequence

(2) $$p^0 \cdot q^0,\ p^0 \cdot q^1,\ p^0 \cdot q^2, \ldots.,\ p^0 \cdot q^T .$$

In practice, the computation is carried out elementwise in two ways. One way is to multiply (inflate) each commodity's nominal period 0 value by its quantity change,

(3) $$p_n^0 q_n^0 ( q_n^t / q_n^0 ) = p_n^0 q_n^t \quad (t = 1,\ldots,T).$$

The other way is to divide (deflate) each commodity's period t value by its price change,

(4) $$p_n^t q_n^t /( p_n^t / p_n^0 ) = p_n^0 q_n^t \quad (t = 1,\ldots,T).$$

The adding-up of $p_n^0 q_n^t$ for $n$ = 1,…,$N$ delivers $p^0 \cdot q^t$ for each period $t$. Recall that the Laspeyres price index is defined by $P_L(t,t') \equiv p^t \cdot q^{t'} / p^{t'} \cdot q^{t'}$, the Paasche price index is defined by $P_P(t,t') \equiv p^t \cdot q^t / p^{t'} \cdot q^t$, the Laspeyres quantity index is defined by $Q_L(t,t') \equiv p^{t'} \cdot q^t / p^{t'} \cdot q^{t'}$, and the Paasche quantity index is defined by $Q_P(t,t') \equiv p^t \cdot q^t / p^t \cdot q^{t'}$. With hindsight, the sequence (2) can be considered as having been obtained by taking the sequence of nominal values given in (1) and deflating these using the Paasche price index numbers, or by inflating these using the Laspeyres quantity index numbers, since

(5) $$p^0 \cdot q^t = p^t \cdot q^t / P_P(t,0) = p^0 \cdot q^0 Q_L(t,0) \quad (t = 1,\ldots,T).$$

The aggregate quantity change between any two periods can now be computed simply by taking the ratio of the corresponding two values from the sequence (2). For instance, if one is interested in the change between two adjacent periods $t$-1 and $t$, this is given by

(6) $$p^0 \cdot q^t / p^0 \cdot q^{t-1} = Q_{Lo}(t,t-1;0) \quad (t = 1,\ldots,T).$$

This formula is an instance of what in the literature is known as a Lowe quantity index[4]. Its interpretation is straightforward: the numerator contains the period $t$ quantities evaluated at their base period prices, and the denominator contains the period $t$-1 quantities evaluated at the same prices.

[3] I associate this view with the *SNA* 1968, the relevant paragraph being 4.46.
[4] Some are accustomed to calling this a 'modified Laspeyres quantity index'.

The framework provided by (1), (2) and (6) has the virtue of simplicity. This simplicity does not carry through, however, to the price index counterpart to the Lowe quantity index given in (6). This price index, which can be obtained by dividing the value change by the quantity change, is given by:

$$(7) \qquad \frac{p^t \cdot q^t / p^{t-1} \cdot q^{t-1}}{p^0 \cdot q^t / p^0 \cdot q^{t-1}} \quad (t = 1,\ldots,T).$$

This formula not only is less simple than (6), but also has an important disadvantage. Suppose that between periods $t$-1 and $t$ all the prices change by the same factor, that is, $p_n^t = \lambda p_n^{t-1}$ ($n$ = 1,…,$N$) for a certain $\lambda > 0$. In this situation, formula (7) in general will exhibit an outcome different from $\lambda$.

In practice one also has to face all the difficulties connected with the fact that our assumption of (an) unchanging (set of) commodities is not valid. First, in the course of time, new commodities enter the aggregate. The problem becomes clear by looking at formulas (3) and (4). For any new commodity, its base period value as well as quantity equal zero; hence, formula (3) cannot be used. Although the period $t$ value and price are known, the base period price does not exist; hence, formula (4) cannot be used either. Of course, for commodities that in the course of time have disappeared from the aggregate an analogous problem.

Second, even when there are no (dis-) appearing commodities, usually it is still necessary to deal with quality change. Quality change of commodity $n$ occurs when its period t price cannot be compared to its base period price without allowing for changes in the nature of the commodity; or, equivalently, when its period $t$ quantity cannot immediately be compared to its base period quantity. Dependent on the calculation method chosen – according to formula (3) or (4) – the quantity or price change must somehow be adjusted for the quality change that has occurred.

The important point is that in all these cases, imputations or estimates must be made, and this becomes more difficult and more dubious the longer the time span becomes between the base period and period $t$. In addition, with the lapse of time it becomes less and less meaningful to aggregate recent quantities with prices from a past period, as in expression (6). Therefore, every five or ten years, a new set of constant prices must be taken to act as base prices, which causes structural breaks in the time series of values-at-constant-prices.

## 3. The Modern Point of View

The modern view is rooted in the perspective that primary interest lies in measuring the real change between two adjacent periods. Stated more formally, according to the modern view, the primary problem is to decompose the value change,

$$(8) \qquad p^t \cdot q^t / p^{t-1} \cdot q^{t-1} \quad (t = 1,\ldots,T),$$

into price and quantity change components. There are various ways to do this. One frequently used approach decomposes the value change into a Paasche price index and a Laspeyres quantity index; that is, the value change is decomposed as:

(9) $$\frac{p^t \cdot q^t}{p^{t-1} \cdot q^{t-1}} = P_P(t,t-1)Q_L(t,t-1) \quad (t = 1,\ldots,T).$$

Alternatively, the axiomatic approach leads to the recommendation[5] to use Fisher price and quantity indices for this decomposition. Using Fisher indices, the value change can be decomposed as follows:

$$\frac{p^t \cdot q^t}{p^{t-1} \cdot q^{t-1}} = \left[\frac{p^t \cdot q^{t-1}}{p^{t-1} \cdot q^{t-1}} \frac{p^t \cdot q^t}{p^{t-1} \cdot q^t}\right]^{1/2} \left[\frac{p^{t-1} \cdot q^t}{p^{t-1} \cdot q^{t-1}} \frac{p^t \cdot q^t}{p^t \cdot q^{t-1}}\right]^{1/2}$$

(10) $$\equiv P_F(t,t-1)Q_F(t,t-1) \quad (t = 1,\ldots,T).$$

The first term in square brackets on the right-hand side of the first equality sign in (10) is the price index and the second one is the quantity index.[6]

New commodities, disappearing commodities, and quality change also cause problems in the computation of the components of (10). However, since the time span between periods *t*-1 and *t* is quite small – usually a year – the extent of the problems that must be solved is smaller than in the case discussed in the previous section: there are fewer new and disappearing commodities, and fewer (and probably smaller) quality changes to account for when comparing two adjacent periods than two periods far apart.

Not so well known, but extremely useful, is the fact that the Fisher quantity index can be written in a form comparable to formula (6). This result, for the first time discovered by Jan van IJzeren (1952), reads

(11) $$Q_F(t,t-1) = \frac{\frac{1}{2}(p^{t-1} + p^t / P_F(t,t-1)) \cdot q^t}{\frac{1}{2}(p^{t-1} + p^t / P_F(t,t-1)) \cdot q^{t-1}} \quad (t = 1,\ldots,T).$$

The numerator contains the period *t* quantities valued at the average, deflated prices for periods *t*-1 and *t*, whereas the denominator contains the period *t*-1 quantities valued at the same deflated prices as appear in the numerator. Notice that each individual component of the price vector $p^{t-1} + p^t / P_F(t,t-1)$ depends on all the prices and all the quantities. This formula enables one to view the measure for the aggregate quantity change, $Q_F(t,t-1)$, as a weighted arithmetic average of individual quantity changes, $q_n^t / q_n^{t-1}$ ($n = 1,\ldots,N$).[7] This makes clear to what extent the various commodities contribute to the aggregate quantity change.

Does there exist in this approach a more general analogue to the sequence of values-at-constant-prices (2)? The answer appears to be: yes. Based on expression (5), the analogue to (2) is given by the sequence of real values

---

[5] A summary of the underlying literature can be found in Diewert (1996).

[6] Because these are Fisher indices, the price and quantity indices have the same functional form; that is, by interchanging prices and quantities the indices transform into each other.

[7] See Balk (2004) for alternatives. Formula (11) has been in use since 1999 by the U.S. Bureau of Economic Analysis; see Ehemann *et al*. (2002). Of course, a similar formula holds for the Fisher price index.

(12) $p^t \cdot q^t / P(t,0) = p^0 \cdot q^0 Q(t,0)$ $(t = 1,\ldots,T)$,

where $P(t,t')$ is some price index and $Q(t,t')$ is some quantity index. Notice that (12) expresses in a slightly different form what in the axiomatic approach is called the Product Test.

The *SNA* 1993 recommends either of two methods. One is to start at the left-hand side of (12) and to deflate nominal values by chained Fisher price index numbers; that is, to replace $P(t,0)$ by

(13) $P_F^c(t,0) \equiv \prod_{\tau=1}^{t} P_F(\tau,\tau-1)$ $(t = 1,\ldots,T)$.

The other is to start at the right-hand side of (12) and to inflate the nominal base period value by chained Fisher quantity index numbers; that is, to replace $Q(t,0)$ by

(14) $Q_F^c(t,0) \equiv \prod_{\tau=1}^{t} Q_F(\tau,\tau-1)$ $(t = 1,\ldots,T)$.

The real values obtained in this manner correspond to what in the United States have come to be called 'chained dollars'[8]. The use of chained Fisher price index numbers in (13) is consistent with (10). This follows because, dividing the real values of two adjacent periods into each other yields

(15) $$\frac{p^t \cdot q^t / P_F^c(t,0)}{p^{t-1} \cdot q^{t-1} / P_F^c(t-1,0)} = \frac{p^t \cdot q^t / p^{t-1} \cdot q^{t-1}}{P_F(t,t-1)} = Q_F(t,t-1) \quad (t = 1,\ldots,T),$$

which is an expression for the quantity change that has occurred between the two periods. The same holds for the use of chained Fisher quantity index numbers as in (14).

An unsatisfactory alternative was proposed by Hillinger (2002); see the Appendix for details.

## 4. Comparison

The traditional approach gives priority to the construction of sequences of values-at-constant-prices according to expression (5). Quantity changes between adjacent periods are then evaluated using expression (6). The modern approach gives priority to the computation of quantity index numbers for adjacent periods according to expression (10). Real values can then be computed using expression (12) and chained index numbers. These are two distinct paradigms.

The core of Von der Lippe's critique (mentioned in the text and footnote 2 in section 1 above) is that the properties of the sequence of real values given in (5) differ from those of (12), and that the properties of the Lowe quantity index given in (6) differ from those of the Fisher

[8] The practice in other countries is to use chained Paasche price index numbers and Laspeyres quantity index numbers respectively, as was recommended by Al *et al*. (1986); see also De Boer *et al*. (1997). The *ESA* 1995 considers this practice to be acceptable. The use of chained Fisher index numbers was already mentioned in the *SNA* 1968, par. 4.47.

quantity index in (10). One important difference is that the real values computed according to (12) by chained index numbers are not additive, whereas the real values in (5) do exhibit additivity. Thus, the chained index numbers (can) exhibit behavior that is different from the direct index numbers.

It is relatively simple to show that the two approaches cannot be unified; that is done in this section.

The first key question is whether there exists a quantity index $Q(t,t')$ such that

(16) $$Q(t,t') = Q(t,0)/Q(t',0).$$

A quantity index that satisfies this condition exhibits the property of circularity and can be written as

(17) $$Q(t,t') = f(t)/f(t').$$

The fundamental requirement that $Q(t,t') = 1$ if the quantity vectors of the two periods are equal leads to the conclusion that *f(t)* in (17) must be a function of the quantities only. Hence, prices cannot play any role in $Q(t,t')$.[9] This implies that the price index corresponding to $Q(t,t')$, $(p^t \cdot q^t / p^{t'} \cdot q^{t'})/Q(t,t')$, does *not* pass the fundamental Identity Test; that is, if the price vectors of the two periods are equal, then this last expression will not necessarily equal 1.

The second key question concerns the additivity, or, more generally, the consistency-in-aggregation, of price and quantity indices. Suppose that our aggregate can be partitioned into K subaggregates and let (after permutation of commodities) the price and quantity vectors be partitioned as $p^t = (p_1^t,...,p_K^t)$ and $q^t = (q_1^t,...,q_K^t)$ respectively, where $(p_k^t, q_k^t)$ is the subvector corresponding to the subaggregate *k* = 1,…,*K*. Let $P_k(t,t')$ be a price index with the same functional form as $P(t,t')$, but with its number of variables reduced to the number of commodities of subaggregate *k*. Similarly, let $Q_k(t,t')$ be a quantity index with the same functional form as $Q(t,t')$, but with its number of variables reduced to the number of commodities of subaggregate *k*. Now the real values computed according to (12) are called *additive* if

(18a) $$\sum_{k=1}^{K} \frac{p_k^t \cdot q_k^t}{P_k(t,0)} = \frac{p^t \cdot q^t}{P(t,0)};$$

or, in other words, if the real subaggregate values add up to the real aggregate value. In terms of quantity indices, additivity means that

(18b) $$\sum_{k=1}^{K} p_k^0 \cdot q_k^0 Q_k(t,0) = p^0 \cdot q^0 Q(t,0).$$

[9] A more formal proof is given by Balk (1995); see also Balk (2008).

The more general concept of consistency-in-aggregation for price and quantity indexes was defined by Balk (1995), (1996), (2008)[10]. A price index $P(t,t')$ is called consistent-in-aggregation if

(19a) $$\sum_{k=1}^{K}\psi(P_k(t,t'),p_k^t\cdot q_k^t,p_k^{t'}\cdot q_k^{t'})=\psi(P(t,t'),p^t\cdot q^t,p^{t'}\cdot q^{t'}),$$

where $\psi(.)$ is a function that is continuous and strictly monotonic in its first variable. Likewise, a quantity index $Q(t,t')$ is called consistent-in-aggregation if

(19b) $$\sum_{k=1}^{K}\zeta(Q_k(t,t'),p_k^t\cdot q_k^t,p_k^{t'}\cdot q_k^{t'})=\zeta(Q(t,t'),p^t\cdot q^t,p^{t'}\cdot q^{t'}),$$

where $\zeta(.)$ is a function that is continuous and strictly monotonic in its first variable.

There are many, in fact infinitely many, functional forms for price and quantity indices that satisfy (19a) or (19b). As an example, the reader is invited to consider the generalized mean price index $P(t,t')=[\sum_{n=1}^{N}(v_n^{t'}/V^{t'})(p_n^t/p_n^{t'})^{\rho}]^{1/\rho}$ where $\rho\neq 0$. However, problems arise as soon as a number of very basic requirements are imposed on the price and quantity indices.

Suppose it is assumed that

- the price and quantity indices satisfy the Product Test (12);
- the price index satisfies the Equality Test; that is, if all the subaggregate price index numbers are equal – that is, if $P_k(t,t')=\lambda$ for all $k$ = 1,…,$K$ – then the aggregate price index number takes on the same magnitude, $P(t,t')=\lambda$;
- the quantity index satisfies the Equality Test; that is, if $Q_k(t,t')=\lambda$ for all $k$ = 1,…,$K$, then $Q(t,t')=\lambda$;
- the price index $P(t,t')$ is linearly homogeneous in current period prices $p^t$;
- when the number of commodities in an aggregate reduces to 1, then the price index reduces to a price relative; that is, $P(t,t')=p^t/p^{t'}$ whenever $N$ = 1.

Under these assumptions it can be shown that the only price indices satisfying the consistency-in-aggregation requirement (19a) are the Laspeyres and Paasche.[11] Moreover, it is straightforward to show that any chained price index deviates from these two functional forms. For instance, for the chained Laspeyres price index it can be shown that

(20) $$P_L^c(t,0)=\prod_{\tau=1}^{t}\frac{p^{\tau}\cdot q^{\tau-1}}{p^{\tau-1}\cdot q^{\tau-1}}=\frac{p^{*}\cdot q^0}{p^0\cdot q^0}\neq\frac{p^t\cdot q^0}{p^0\cdot q^0},$$

since

[10] Pursiainen (2005) proposed a more general definition of consistency-in-aggregation, which appears to reduce to the one presented here for the situations considered here.
[11] See Balk (1995), (2008).

(21) $$p^* = p^1 \prod_{\tau=2}^{t} \frac{p^\tau \cdot q^{\tau-1}}{p^{\tau-1} \cdot q^{\tau-1}} \neq p^t .$$

Given this mathematical, state-of-affairs, it seems justified that priority is given to decomposing the value change between adjacent periods into price and quantity index components. If one is to construct real values for a sequence of periods, then chained index numbers must be used for deflating or inflating.[12] With the present day computation facilities and the basic data, however, it should be relatively simple, for analytical purposes, to compute alternative price and quantity index numbers, as well as alternative sequences of real values, among which are included values-at-constant-prices.

## 5. On the Economic Theoretic Interpretation of Chained Index Numbers

The strategy of chaining has primarily been motivated by practical considerations. The question considered in this, and the next, section is: what precisely does a chained index measure? This section approaches the question from the economic-theoretic point of view. For direct (bilateral) price and quantity indexes there is a well-established body of theory. Can this theory be used to provide an answer to our question? That question is addressed here.

### 5.1 Constant homothetic preference ordering

Suppose our price and quantity data $(p^t, q^t)$ for $t$ = 0,1,…,$T$ can be rationalized by a utility function. That is, suppose there exists a continuous function $U(q)$ representing a preference ordering that satisfies mild regularity conditions. More specifically, suppose that

(22) $$p^t \cdot q^t = C(p^t, U(q^t)),$$

where $C(p,u) \equiv \min_q \{p \cdot q \,|\, U(q) \geq u\}$ is the cost function that is dual to $U(q)$. Duality theory tells us that U(q) is homothetic if and only if the cost function can be decomposed as

(23) $$C(p,u) = F(u)C(p,1) \equiv F(u)c(p),$$

where $F(u)$ is a function that is monotonicly increasing in $u$, and $c(p)$ is called the unit cost function. Varian (1983), based on earlier work by Diewert (1973), showed that there exists a data rationalizing utility function, which is homothetic if and only if a condition called the Homothetic Axiom of Revealed Preference (HARP) is satisfied. The specific form of this function is of no concern here.

As is well known, the Konüs cost of living index for period $t$ relative to period $t'$, conditional on the utility level $u$, is defined by

(24) $$P_K(t,t';u) \equiv \frac{C(p^t,u)}{C(p^{t'},u)} \qquad u \in \text{Range}(U).$$

[12] A practical way of dealing with the additivity problem was developed by Balk and Reich (2008).

If the utility function is homothetic, then the Konüs cost of living index can be expressed as the ratio of values of the unit cost function; that is, the Konüs cost of living index can be expressed as

(25) $$P_K(t,t';u) = c(p^t)/c(p^{t'}) \equiv P_K(t,t')$$

for any two periods $t$, $t'$. Using relations (25), (22), and the definition of the cost function, it is straightforward to derive the well-known Laspeyres and Paasche bounds:

(26) $$P_K(t,t') = \frac{C(p^t,U(q^{t'}))}{C(p^{t'},U(q^{t'}))} \le \frac{p^t \cdot q^{t'}}{p^{t'} \cdot q^{t'}} = P_L(t,t')$$

(27) $$P_K(t,t') = \frac{C(p^t,U(q^t))}{C(p^{t'},U(q^t))} \ge \frac{p^t \cdot q^t}{p^{t'} \cdot q^t} = P_P(t,t').$$

Based on this double inequality, it is reasonable to view the Fisher price index, $P_F(t,t') = [P_L(t,t')P_P(t,t')]^{1/2}$, as an approximation to the Konüs index $P_K(t,t')$. In fact, $P_F(t,t') = P_K(t,t')$ if and only if the unit cost function $c(p)$ is quadratic.[13]

However, many other sets of bounds can also be derived. Consider for instance an arbitrary third period $0 \le s \le T$. Then, by the same method, we find that also

(28) $$P_K(t,t') = \frac{c(p^t)}{c(p^s)}\frac{c(p^s)}{c(p^{t'})} \le \frac{p^t \cdot q^s}{p^s \cdot q^s}\frac{p^s \cdot q^{t'}}{p^{t'} \cdot q^{t'}} = P_L(t,s)P_L(s,t'),$$

and

(29) $$P_K(t,t') = \frac{c(p^t)}{c(p^s)}\frac{c(p^s)}{c(p^{t'})} \ge \frac{p^t \cdot q^t}{p^s \cdot q^t}\frac{p^s \cdot q^s}{p^{t'} \cdot q^s} = P_P(t,s)P_P(s,t').$$

The obvious generalization of the above procedure is to consider all spanning trees connecting the periods 0,1,…,$T$. A spanning tree is a connected graph without cycles. Suppose that on such a tree the periods $t'$ and $t$ are connected via the periods $s(2),\ldots,s(L\text{-}1)$, where $L \ge 3$, and call $t' = s(1)$ and $t = s(L)$. Let $L$=2 represent the case where $t'$ and $t$ are adjacent (hence the number of intermediate periods equals zero). Then

(30) $$P_K(t,t') = \prod_{\ell=2}^{L} \frac{c(p^{s(\ell)})}{c(p^{s(\ell-1)})} \le \prod_{\ell=2}^{L} P_L(s(\ell),s(\ell-1)).$$

Taking the minimum of the right-hand side of this expression over all spanning trees delivers the tightest upper bound for $P_K(t,t')$. Similarly, one obtains that

(31) $$P_K(t,t') = \prod_{\ell=2}^{L} \frac{c(p^{s(\ell)})}{c(p^{s(\ell-1)})} \ge \prod_{\ell=2}^{L} P_P(s(\ell),s(\ell-1)),$$

[13] See Konüs and Byushgens (1926), Diewert (1976), and Lau (1979).

and taking the maximum of the right-hand side of this expression over all spanning trees delivers the tightest lower bound for $P_K(t,t')$. Both of these tightest bounds can be computed by employing Warshall's algorithm. This algorithm also checks whether HARP is satisfied and, if so, computes the tightest upper and lower bounds.

It is clear that, given that HARP is satisfied, the (direct) Laspeyres price index $P_L(t,t')$ as well as the chained Laspeyres price index $P_L^c(t,t')$ are elements of the set of upper bounds for the Konüs cost of living index $P_K(t,t')$. Similarly, the (direct) Paasche price index $P_P(t,t')$ as well as the chained Paasche price index $P_P^c(t,t')$ are elements of the set of lower bounds. If $P_L^c(t,t') < P_L(t,t')$ then the chained Laspeyres price index is a tighter upper bound for the Konüs index than the (direct) Laspeyres price index. Similarly, if $P_P^c(t,t') > P_P(t,t')$ then the chained Paasche price index is a tighter lower bound for the Konüs index than the (direct) Paasche price index.

We may conclude that, if both conditions are satisfied, then the chained Fisher price index $P_F^c(t,t') = [P_L^c(t,t')P_P^c(t,t')]^{1/2}$ is a better approximation to $P_K(t,t')$ than the (direct) Fisher price index.

## 5.2 Constant preference ordering

However, the nice result just derived only holds when HARP is satisfied. When HARP is not satisfied, it is still possible that there exists a data rationalizing utility function such that (22) holds; however, this function is not necessarily homothetic. Varian (1982), based on earlier work by Afriat and Diewert (1973), showed this to be the case if and only if a condition called the Generalized Axiom of Revealed Preference (GARP) is satisfied. Under this weaker assumption, the standard bounding result reads:

(32) $$P_K(t,t';U(q^{t'})) \leq P_L(t,t')$$

(33) $$P_K(t,t';U(q^t)) \geq P_P(t,t').$$

It can then be shown[14] that there exists a utility level $u^*$ between $U(q^{t'})$ and $U(q^t)$ such that $P_K(t,t';u^*)$ lies between $P_L(t,t')$ and $P_P(t,t')$. $P_F(t,t')$ is a symmetric average of $P_L(t,t')$ and $P_P(t,t')$. Hence, if the interval between $P_L(t,t')$ and $P_P(t,t')$ is small, it would be expected that

(34) $$P_F(t,t') \approx P_K(t,t';u^*) \text{ for some } u^* \text{ between } U(q^{t'}) \text{ and } U(q^t).$$

The result given above is interesting, but not very useful if the periods $t$ and $t'$ are far apart and the difference between the Laspeyres and Paasche price index numbers is large. If this is the case, it may be better to consider the chained Fisher price index, which is built up from

[14] The proof by Diewert (1981) goes back to Konüs.

comparisons of adjacent periods. For these comparisons, the Laspeyres-Paasche spread may be more likely to be small enough to justify the use of (34). Hence,

(35) $$P_F^c(t,t') = \prod_{\tau=t'+1}^{t} P_F(\tau,\tau-1) \approx \prod_{\tau=t'+1}^{t} P_K(\tau,\tau-1;u^{\tau*})$$

for some $u^{\tau*}$ between $U(q^{\tau-1})$ and $U(q^{\tau})$.

This result is still not very insightful. Equation (35) means that the chained Fisher price index approximates a chained Konüs index where the levels of utility vary over time. Getting rid of the variation in levels of utility would be helpful. This can be accomplished by noticing that the Konüs index defined in (24) is continuous in the utility level *u*. Choose $s = (t+t'+1)/2$ and assume that

(36) $$P_K(\tau,\tau-1;u^{\tau*}) = P_K(\tau,\tau-1;U(q^s))\exp\{a(\tau-s)\} \text{ for some } a \neq 0,$$

which means that, conditional on prices $p^{\tau}$ and $p^{\tau-1}$, $P_K(\tau,\tau-1;u^{\tau*})$ is a loglinear function of the time variable associated with the reference utility level. By elementary analytical methods, one can then show that

(37) $$\prod_{\tau=t'+1}^{t} P_K(\tau,\tau-1;u^{\tau*}) = \prod_{\tau=t'+1}^{t} P_K(\tau,\tau-1;U(q^s)) = P_K(t,t';U(q^s)),$$

where the last equality follows from the transitivity of the Konüs index for some fixed *u*. Thus, if (36) holds, then the chained Fisher price index $P_F^c(t,t')$ can be viewed as approximating the Konüs cost of living index $P_K(t,t';U(q^s))$, where s is an intermediate time period. Notice that assumption (36) rules out any cycles.

## 5.3 Variable preference ordering

A still weaker, but not testable, assumption is that the preference ordering is changing over time, so that (22) must be replaced by

(38) $$p^t \cdot q^t = C^t(p^t, U^t(q^t))$$

where $U^t(q)$ represents the period *t* preference ordering and $C^t(p,u)$ represents the period *t* dual cost function. The Laspeyres and Paasche bounds still apply, but must be reformulated as

(39) $$P_K^{t'}(t,t';U^{t'}(q^{t'})) \leq P_L(t,t')$$

(40) $$P_K^{t}(t,t';U^{t}(q^{t})) \geq P_P(t,t').$$

A result such as (34), however, is now impossible because the utility functions $U^{t'}(q)$ and $U^t(q)$ represent different preference orderings. It is meaningless to compare their numerical values across periods. There is a way out, however. A cost of living index including the preference change effect was defined by Balk (1989) as

(41) $$P^{t,t'}(t,t';q) \equiv \frac{C^t(p^t,U^t(q))}{C^{t'}(p^{t'},U^{t'}(q))}.$$

This index conditions on the quantity vector $q$ and compares the period $t$ cost of the period $t$ indifference class of $q$ to the period $t'$ cost of the period $t'$ indifference class of $q$. It is a natural extension of the Konüs cost of living index: if the period $t$ and $t'$ preference orderings are identical, then $P^{t,t'}(t,t';q) = P_K(t,t';U(q))$. The index (41) can be decomposed into two parts relating, respectively, to the effects of price change and preference change. The effect of preference change is measured by $P^{t,t'}(t,t';q)$ by setting $p^t = p^{t'}$. This effect is not necessarily equal to 1, but, as argued by Balk (1989), has the right sign.

Balk (1989) also showed that the Laspeyres and Paasche bounds still apply, so that:

(42) $$P^{t,t'}(t,t';q^{t'}) \le P_L(t,t')$$

(43) $$P^{t,t'}(t,t';q^t) \ge P_P(t,t').$$

In this case, Diewert's (1981) proof can be used to show that there exists a quantity vector $q^*$ between $q^{t'}$ and $q^t$ such that $P^{t,t'}(t,t';q^*)$ lies between $P_L(t,t')$ and $P_P(t,t')$. If the interval between $P_L(t,t')$ and $P_P(t,t')$ is small, then one may expect the following result to hold for the Fisher price index:

(44) $P_F(t,t') \approx P^{t,t'}(t,t';q^*)$ for some $q^*$ between $q^{t'}$ and $q^t$.

Assuming now that for adjacent periods the Laspeyres-Paasche spread is indeed small, the following approximation using the chained Fisher price index may be close enough to be useful:

(45) $$P_F^C(t,t') = \prod_{\tau=t'+1}^{t} P_F(\tau,\tau-1) \approx \prod_{\tau=t'+1}^{t} P^{\tau,\tau-1}(\tau,\tau-1;q^{\tau*})$$

for some $q^{\tau*}$ between $q^{\tau-1}$ and $q^\tau$.

The right-hand side of expression (45) contains indices that are conditional on quantity vectors that vary through time. Using the continuity of $P^{\tau,\tau-1}(\tau,\tau-1;q)$ in $q$, we assume that

(46) $$P^{\tau,\tau-1}(\tau,\tau-1;q^{\tau*}) = P^{\tau,\tau-1}(\tau,\tau-1;q^s)\exp\{b(\tau-s)\} \text{ for some } b \ne 0;$$

that is, conditional on prices $p^\tau$ and $p^{\tau-1}$, $P^{\tau,\tau-1}(\tau,\tau-1;q^{\tau*})$ is a loglinear function of the time variable associated with the reference quantity vector. Then, as in the previous subsection, it can be shown that

(47) $$\prod_{\tau=t'+1}^{t} P^{\tau,\tau-1}(\tau,\tau-1;q^{\tau*}) = \prod_{\tau=t'+1}^{t} P^{\tau,\tau-1}(\tau,\tau-1;q^s) = P^{t,t'}(t,t';q^s),$$

where the last equality follows from the transitivity of (41) for fixed *q*. Thus, if (46) holds, the chained Fisher price index $P_F^C(t,t')$ may be considered to provide an approximation to the cost of living index including the preference change effect $P^{t,t'}(t,t';q^s)$, where *s* is an intermediate time period. Notice that assumption (46) also rules out any cycles.

Recall that $P^{t,t'}(t,t';q^s)$ is not necessarily equal to 1 when $p^t = p^{t'}$. This feature is shared by a chained index such as $P_F^C(t,t')$. Put otherwise, the fact that a chained index violates the (bilateral) Identity Test reflects the fact that such an index encompasses the effect of preference change.

## 6. A Divisia Index Theory Perspective

For those who do not believe in well-behaved preference orderings and optimization, Divisia index theory might be used to shed light on the relation between direct and chained indices. This theory, however, requires a mental leap: time periods must be considered as being of infinitesimal length and time itself as a continuous variable. Prices and quantities are supposed to be strictly positive, continuous and piecewise differentiable functions of time. Thus, when time $\tau$ moves from period 0 to period *T*, prices and quantities $\langle p(\tau),q(\tau)\rangle$ map out a path through the 2N-dimensional, strictly positive, Euclidean orthant. It is also assumed that observations are available at periods 0, 1, 2, …, *T*; that is, it is assumed that we observe

(48) $$p(\tau)=p^{\tau} \text{ and } q(\tau)=q^{\tau} \text{ for } \tau = 0,1,...,T .$$

The starting point for Divisia index theory is the Product Test equation (12). It is straightforward to show, using elementary integral calculus, that this equation can be written as

(49) $$p^t \cdot q^t / p^0 \cdot q^0 = p(t)\cdot q(t)/ p(0)\cdot q(0) = P^{Div}(t,0)Q^{Div}(t,0) \quad (t = 1,\dots,T)$$

where

(50) $$\ln P^{Div}(t,0) \equiv \int_{\tau=0}^{t}\sum_{n=1}^{N} s_n(\tau)\frac{d\ln p_n(\tau)}{d\tau}d\tau ,$$

(51) $$\ln Q^{Div}(t,0) \equiv \int_{\tau=0}^{t}\sum_{n=1}^{N} s_n(\tau)\frac{d\ln q_n(\tau)}{d\tau}d\tau ,$$

and

(52) $$s_n(\tau) \equiv p_n(\tau)q_n(\tau)/ p(\tau)\cdot q(\tau) \quad (n = 1,\dots,N).$$

The problem is how to estimate these index numbers, given that one only has observations on prices and quantities for a finite number of periods. Integral calculus provides us with the following two useful decompositions:

(53) $$P^{Div}(t,0) \equiv \prod_{\tau=1}^{t} P^{Div}(\tau,\tau-1) \quad (t = 1,\dots,T)$$

and

(54) $$Q^{Div}(t,0) \equiv \prod_{\tau=1}^{t} Q^{Div}(\tau,\tau-1) \quad (t = 1,\ldots,T).$$

Now, as demonstrated by Balk (2005), (2008, Chapter 6), for any pair of bilateral price and quantity indices $\langle P(t,t'),Q(t,t')\rangle$ there exists a (hypothetical) vector of functions $C \equiv \langle \hat{p}(\tau),\hat{q}(\tau)\rangle$, defined over the interval $[t',t]$ such that $\hat{p}(t') = p(t')$, $\hat{q}(t') = q(t')$, $\hat{p}(t) = p(t)$, and $\hat{q}(t) = q(t)$, and such that

(55) $$P(t,t') = P_C^{Div}(t,t')$$

(56) $$Q(t,t') = Q_C^{Div}(t,t'),$$

where the subscript $C$ indicates that the integrals are computed using the functions defined by C rather than the true, but unknown, functions occurring in (50) and (51). The closer one believes that $C$ approximates these unknown functions, the better $\langle P(t,t'),Q(t,t')\rangle$ will approximate $\langle P^{Div}(t,t'),Q^{Div}(t,t')\rangle$. The survey quoted makes clear as well that $\langle P_F(t,t'),Q_F(t,t')\rangle$ corresponds to a more reasonable price-quantity path than, say, $\langle P_P(t,t'),Q_L(t,t')\rangle$.

Given this theoretical knowledge, there are two distinct ways of approximating $\langle P^{Div}(t,0),Q^{Div}(t,0)\rangle$. The first is by calculating direct index numbers $\langle P_F(t,0),Q_F(t,0)\rangle$, which use only the period 0 and $t$ data and map out a path over the whole time interval. The second is, according to expressions (53) and (54), by calculating chained index numbers $\langle P_F^c(t,0),Q_F^c(t,0)\rangle$. These chained index numbers also use the available data for the intermediate periods and map out a segmented path that coincides with the true one at the observation points. It seems clear that this second option should be preferred, since all available observations are used this way and the hypothesized path will stay closer to the true one.

## 7. Conclusion

By way of conclusion I return to the main problem: that of decomposing a value ratio into price and quantity components. Let $\langle P(t,t'),Q(t,t')\rangle$ be a pair of bilateral price and quantity indices that satisfy the Product Test. Then we have for any period t = 2, …, $T$ the choice between the decompositions

(57) $$V^t/V^0 = P(t,0)Q(t,0)$$

and

$$V^t/V^0 = \prod_{\tau=1}^{t} V^{\tau}/V^{\tau-1} = \prod_{\tau=1}^{t} P(\tau,\tau-1)Q(\tau,\tau-1)$$

(58) $$= \prod_{\tau=1}^{t} P(\tau,\tau-1) \prod_{\tau=1}^{t} Q(\tau,\tau-1);$$

that is, we have the choice between using direct indices or chained indices. Notice, however, that expression (57) can easily be rewritten as

(59) $$V^t / V^0 = \left[ \prod_{\tau=1}^{t} \frac{P(\tau,0)}{P(\tau-1,0)} \right] \left[ \prod_{\tau=1}^{t} \frac{Q(\tau,0)}{Q(\tau-1,0)} \right],$$

the form of which is comparable to that of expression (58). From this point of view, the question is not so much whether to decompose the value ratio between periods *t* and 0 by direct or chained indices, but whether adjacent periods should be compared by indices of the form $\langle P(\tau,0)/P(\tau-1,0), Q(\tau,0)/Q(\tau-1,0) \rangle$ or $\langle P(\tau,\tau-1), Q(\tau,\tau-1) \rangle$. Posed in this way, the answer seems obvious, because it is not at all clear why period 0 price and/or quantity data should play a role in the comparison of periods $\tau$ and $\tau-1$ ($\tau$ = 2, …, $t$).

As advanced in section 5.3, micro-economic theory suggests the use of Fisher indices for the comparison of adjacent periods, since in that case the chained price and quantity indices admit the respective interpretation of being approximations to cost of living and standard of living indices under changing preferences. The main condition on which this result is predicated is that the observed quantities do not exhibit cyclical behavior.

## Appendix: A Note on Hillinger's (2002) Proposal

Hillinger (2002) proposed to replace chained Fisher price index numbers by chained Marshall-Edgeworth price index numbers; that is, he proposed to replace formula (13) by

(A.1) $$P_{ME}^{c}(t,0) \equiv \prod_{\tau=1}^{t} P_{ME}(\tau,\tau-1) \quad (t = 1,\ldots,T),$$

where the Marshall-Edgeworth price index is defined as

(A.2) $$P_{ME}(t,t') = \frac{\frac{1}{2}(q^{t'} + q^{t}) \cdot p^{t}}{\frac{1}{2}(q^{t'} + q^{t}) \cdot p^{t'}} \quad (t = 1,\ldots,T).$$

This proposal has the disadvantage that the equality of deflation and inflation – see expression (12) – gets lost, since

(A.3) $$\frac{p^t \cdot q^t / p^0 \cdot q^0}{P_{ME}^{c}(t,0)} \neq Q_{ME}^{c}(t,0)$$

where $Q_{ME}^{c}(t,0)$ is a chained Marshall-Edgeworth quantity index defined by (A.1) and (A.2) after interchanging prices and quantities.

It appears that for two adjacent periods the quantity component, (A.4) $$\frac{p^t \cdot q^t / p^{t-1} \cdot q^{t-1}}{P_{ME}(t,t-1)} = \frac{1+Q_L(t,t-1)}{1+1/Q_P(t,t-1)},$$

is dual to the "true factorial price index" and has the disadvantage of being not linearly homogeneous in $q^t$. By mimicking the proof of Balk (1983), it is straightforward to show that the quantity index (A.4) is exact for a linear utility function.

Interestingly, the difference of two real values can be rewritten as

(A.5) $$\frac{p^t \cdot q^t}{P^c_{ME}(t,0)} - \frac{p^{t-1} \cdot q^{t-1}}{P^c_{ME}(t-1,0)}$$

$$= \frac{1}{P^c_{ME}(t-1,0)}\left[\frac{1}{2}\left(\frac{p^t}{P_{ME}(t,t-1)} + p^{t-1}\right)\cdot\left(q^t - q^{t-1}\right) + \frac{1}{2}\left(q^t + q^{t-1}\right)\cdot\left(\frac{p^t}{P_{ME}(t,t-1)} - p^{t-1}\right)\right]$$

$$= \frac{1}{P^c_{ME}(t-1,0)}\left[\frac{1}{2}\left(\frac{p^t}{P_{ME}(t,t-1)} + p^{t-1}\right)\cdot\left(q^t - q^{t-1}\right)\right]$$

$$= \frac{1}{2}\left(\frac{p^t}{P^c_{ME}(t,0)} + \frac{p^{t-1}}{P^c_{ME}(t-1,0)}\right)\cdot\left(q^t - q^{t-1}\right),$$

where the next to last equality is based on definition (A.2). The difference of two real values can thus be written as a weighted average of individual quantity differences, $q^t_n - q^{t-1}_n$, which provides a nice interpretation.

The second component of Hillinger's (2002) proposal is to also use the deflator (A.1) for the computation of real values of subaggregates. The additivity problem is thereby not solved, but circumvented. Hillinger's argument is, however, not convincing. As Ehemann *et al*. (2002) see it, Hillinger's proposal "appears to provide data users with very little information beyond what is already provided in the aggregates valued at current prices.". These authors also show that the Hillinger proposal can lead to perverse outcomes.

## References


Al, P.G., B.M. Balk, S. de Boer and G.P. den Bakker (1986), "The Use of Chain Indices for Deflating the National Accounts", *Statistical Journal of the United Nations ECE* 4, 347-368.

Balk, B.M. (1983), "A Note on the True Factorial Price Index", *Statistische Hefte / Statistical Papers* 24, 69-72.

Balk, B.M. (1989), "Changing Consumer Preferences and the Cost-of-Living Index: Theory and Nonparametric Expressions", *Journal of Economics* 50, 157-169.

Balk, B.M. (1995), "Axiomatic Price Index Theory: A Survey", *International Statistical Review* 63, 69-93.

Balk, B.M. (1996), "Consistency-in-Aggregation and Stuvel Indices", *Review of Income and Wealth* 42, 353-363.

Balk, B.M. (2004), "Decompositions of Fisher Indexes", *Economics Letters* 82, 107-113.

Balk, B.M. (2005), "Divisia Price and Quantity Indices: 80 Years After", *Statistica Neerlandica* 59, 119-158.

Balk, B.M. (2008), *Price and Quantity Index Numbers: Models for Measuring Aggregate Change and Difference*, Cambridge University Press.

Balk, B.M. and U.P. Reich (2008), “Additivity of National Accounts reconsidered,” *Journal of Economic and Social Measurement* 33, 165-178.
Boer, S. de, J. van Dalen and P. Verbiest (1997), Chain Indices in the National Accounts: The Dutch Experience, National Accounts Occasional Paper Nr. NA-087, Statistics Netherlands.
CPI Manual (2004), *Consumer Price Index Manual: Theory and Practice*, International Labour Office, International Monetary Fund, Organisation for Economic Co-operation and Development, Eurostat, United Nations, The World Bank.
Diewert, W.E. (1973), “Afriat and Revealed Preference Theory,” *Review of Economic Studies* 40, 419-425.
Diewert, W.E. (1976), “Exact and Superlative Index Numbers”, “Exact and Superlative Index Numbers,” *Journal of Econometrics* 4, 115-145; reprinted as chapter 8 in W.E. Diewert and A.O. Nakamura (eds.) (1993, pp. 223-252 ), North-Holland.
Diewert, W.E. (1981), “The Economic Theory of Index Numbers: A Survey”, in A. Deaton (ed.), *Essays in the Theory and Measurement of Consumer Behaviour in Honour of Sir Richard Stone*, Cambridge University Press, pp. 163-208; reprinted as chapter 7 in Diewert and Nakamura (1993, pp. 177-221).
Diewert, W.E. (1996), “Price and Volume Measures in the System of National Accounts”, in *The New System of National Accounts*, edited by J. Kendrick, Kluwer Academic Publishers.
Diewert, W. E. and A. O. Nakamura (1993), Essays in Index Number Theory, Vol. I, North-Holland.
Ehemann, C., A.J. Katz and B.R. Moulton (2002), “The Chain-Additivity Issue and the US National Economic Accounts”, *Journal of Economic and Social Measurement* 28, 37-49.
ESA (1995), *European System of Accounts* (Eurostat, 1996).
Hillinger, C. (2002), “Consistent Aggregation and Chaining of Price and Quantity Measures”, *Journal of Economic and Social Measurement* 28, 1-20.
Konüs, A.A. and S.S. Byushgens (1926), “On the Problem of the Purchasing Power of Money” (in Russian), *Voprosi Konyunkturi* II(1), 151-172.
Lau, L.J. (1979), “On Exact Index Numbers”, *The Review of Economics and Statistics* 61, 73-82.
Lehr, J. (1885), *Beiträge zur Statistik der Preise* (J. D. Sauerlander, Frankfurt).
Lippe, P. von der (2000), “Der Unsinn von Kettenindizes”, *Allgemeines Statistisches Archiv* 84, 67-82.
Lippe, P. von der (2001a), *Chain Indices: A Study in Price Index Theory.* Spectrum of Federal Statistics, Volume 16 (Metzler-Poeschel, Stuttgart).
Lippe, P. von der (2001b), “Zur Interpretation von Kettenindizes: Wie U. P. Reich Widersinn bei Kettenindizes zum Verschwinden bringt”, *Allgemeines Statistisches Archiv* 85, 343-347.
Lippe, P. von der (2007), *Index Theory and Price Statistics*, Peter Lang, Frankfurt am Main.
Marshall, A. (1887), “Remedies for Fluctuations of General Prices”, *Contemporary Review* 51, 355-375.
Pursiainen, H. (2005), Consistent Aggregation Methods and Index Number Theory, Research Report No. 106, Department of Economics, University of Helsinki.
Rainer, A. (2002), “Verkettung gegensätzlicher Positionen zu Indexfragen”, *Allgemeines Statistisches Archiv* 86, 385-389.
Reich, U.P. (2000), “Messung des Geldwertes: Zur Statistik und Theorie der reinen Preisbewegung,” *Allgemeines Statistisches Archiv* 84, 461-478.
SNA (1968), *A System of National Accounts*, Studies in Methods, Series F, No. 2, Rev. 3, Department of Economic and Social Affairs, Statistical Office of the United Nations, New York.
SNA (1993), *System of National Accounts 1993*, Prepared under the auspices of the Inter-Secretariat Working Group on National Accounts (Commission of the European Communities, International Monetary Fund, Organisation for Economic Co-operation and Development, United Nations, Worldbank).
Varian, H.R. (1982), “The Nonparametric Approach to Demand Analysis”, *Econometrica* 50, 945-973.
Varian, H.R. (1983), “Nonparametric Tests of Consumer Behaviour”, *Review of Economic Studies* 50, 99-110.
IJzeren, J. van (1952), “On the Plausibility of Fisher’s Ideal Indices” (in Dutch), *Statistische en Econometrische Onderzoekingen (CBS)*, Nieuwe Reeks, 7, 104-115.