Chapter 12

ALTERNATIVE APPROACHES TO INDEX NUMBER THEORY

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1. Introduction

The present paper reconsiders the fundamental concepts of true and exact indexes, as these concepts are defined in the index number literature. These concepts form the bedrock of the economic approach to index number theory. A true index is the underlying target – the thing we are trying to measure. An empirically calculable index is exact when, under certain conditions, it exactly equals the true index. Also discussed briefly is the fundamental distinction between the axiomatic and economic approaches.

This paper was inspired by the 2008 *American Economic Review* paper of Van Veelen and van der Weide, henceforth VW. VW provide some interesting new perspectives on these issues.

VW have two main objectives. First, they attempt to give precise meanings to the concepts of exact and true indexes. A few definitions of a true index have been provided in the literature. VW propose some new and broader definitions that aim to include all of these as special cases. Some of the existing definitions, however, are more established than others. In particular, a broad consensus is already established in favor of the Konüs (1924) and Allen (1949) index definitions (which are closely related). One problem with VW's new definitions are that by seeking to embrace also the less established definition associated with Afriat (1981), they end up with outcomes that are quite abstract and differ considerably from the consensus position. Hence it might have been better if VW had introduced a new terminology rather than adding to the existing definitions of true indexes. VW also identify some problems with the standard definition of exactness, most notably that for some well known index number formulae the exactness property does not always hold for all strictly positive prices. This is an important finding. However, rather than changing the definition of exactness, we argue that what is required is a more careful analysis of the regularity region of exact indexes.

Second, VW reinterpret the distinction between the axiomatic and economic approaches. Their findings rely on the perceived limitations of the economic approach. In our opinion their reinterpretation is problematic. In our view, the economic approach is more flexible than the

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analysis of VW suggests, thus potentially invalidating their demarcations between the two approaches.

Nevertheless, even though we disagree with some of their conclusions, VW's method is novel and raises a number of issues relating to fundamental concepts of index number theory that deserve closer scrutiny. The differences distinguishing the various approaches are explained in the present paper in the context of earlier work of others.

2. Existing Definitions of True Indexes

The first concept of a true index was introduced into the literature in the price index context by Konüs (1924). The theory assumes that a consumer has well defined *preferences* over different combinations of N consumer commodities or items. The consumer's preferences over alternative possible nonnegative consumption vectors $\mathbf{q}$ are assumed to be representable by a nonnegative, continuous, increasing and concave utility function $U(\mathbf{q})$. It is further assumed that the consumer minimizes the cost of achieving the period t utility level $u^i \equiv U(\mathbf{q}^i)$ for periods (or situations) $i = 1,2$. Thus it is assumed that the observed (nonzero) period i consumption vector $\mathbf{q}^1$ solves the following *period i cost minimization problem*:

(1) $$C(u^i, p^i) \equiv \min_q \{\mathbf{p}^i\mathbf{q} : U(\mathbf{q}) = u^i \equiv U(\mathbf{q}^i)\} = \mathbf{p}^i\mathbf{q}^i;\ i = 1,2$$

where the period t price vector $\mathbf{p}^i$ is strictly positive for $i = 1,2$ and $p^iq^i \equiv \sum_{n=1}^N p_n^i q_n^i$.

The Konüs (1924) family of *true cost of living indexes*, pertaining to two periods where the consumer faces the strictly positive price vectors $\mathbf{p}^0$ and $\mathbf{p}^1$ in periods 0 and 1 respectively, is defined as the ratio of the minimum costs of achieving the same utility level $u \equiv U(\mathbf{q})$ where $\mathbf{q}$ is a positive reference quantity vector. Thus, the Konüs true cost of living index with reference quantity vector $\mathbf{q}$ is defined as follows:

(2) $$P_K(\mathbf{p}^1, \mathbf{p}^2, \mathbf{q}) \equiv C[U(\mathbf{q}), \mathbf{p}^2] / C[U(\mathbf{q}), \mathbf{p}^1].$$

We say that definition (2) defines a *family* of true price indexes because there is one such index for each reference quantity vector $\mathbf{q}$ chosen.

If the utility function U happens to be linearly homogeneous (or can be monotonically transformed into a linearly homogeneous function[2]), then definition (2) simplifies to[3]

(3) $$P_K(\mathbf{p}^1, \mathbf{p}^2, \mathbf{q}) = \{U(\mathbf{q})C[1, \mathbf{p}^2]\} / \{U(\mathbf{q})C[1, \mathbf{p}^1]\} = c(\mathbf{p}^2) / c(\mathbf{p}^1),$$

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[2] Shephard (1953) defined a *homothetic* function to be a monotonic transformation of a linearly homogeneous function. However, if a consumer's utility function is homothetic, we can always rescale it to be linearly homogeneous without changing consumer behavior. Hence in what follows, we will simply identify the homothetic preferences assumption with the linear homogeneity assumption.

[3] See Afriat (1972) or Pollak (1983).

where $c(\mathbf{p}^i)$ is the unit cost function $C(1,\mathbf{p}^i)$. Thus in the case of homothetic preferences, the family of true cost of living indexes collapses to a unit cost or expenditure ratio.

The second concept of a true index is the *Allen* (1949) *family of true quantity indexes*, which also uses the consumer's cost or expenditure function in order to define these true indexes. Again, it is assumed that the consumer engages in cost minimizing behavior in each period so that assumptions (1) hold. For each choice of a strictly positive reference price vector $\mathbf{p}$, the Allen true quantity index, $Q_A(\mathbf{q}^1,\mathbf{q}^2,\mathbf{p})$ is defined as

(4) $$Q_A(\mathbf{q}^1,\mathbf{q}^2,\mathbf{p}) \equiv C[U(\mathbf{q}^2),\mathbf{p}]/C[U(\mathbf{q}^1),\mathbf{p}].$$

The basic idea of the Allen quantity index dates back to Hicks (1942) who observed that if the price vector $\mathbf{p}$ were held fixed and the quantity vector $\mathbf{q}$ is free to vary, then $C[U(\mathbf{q}),\mathbf{p}]$ is a perfectly valid cardinal measure of utility.[4]

As with the true Konüs cost of living, the Allen definition simplifies considerably if the utility function happens to be linearly homogeneous. In this case, (4) simplifies to:[5]

(5) $$Q_A(\mathbf{q}^1,\mathbf{q}^2,\mathbf{p}) = \{U(\mathbf{q}^2)C[1,\mathbf{p}]\}/\{U(\mathbf{q}^1)C[1,\mathbf{p}]\} = U(\mathbf{q}^2)/U(\mathbf{q}^1).$$

Thus in the case of homothetic preferences (where preferences can be represented by a linearly homogeneous utility function), the family of Allen quantity indexes collapses to the utility ratio between the two situations.

Note that in the homothetic preferences case, the Allen quantity aggregate for the vector $\mathbf{q}$ is simply the utility level $U(\mathbf{q})$ and the Konüs price aggregate for the price vector $\mathbf{p}$ is the unit cost or expenditure $c(\mathbf{p})$.[6]

A third concept for a true index that appears frequently in the literature is the *Malmquist* (1953) *quantity index*. This index can be defined using only the consumer's utility function $U(\mathbf{q})$ but we will not study this index in any detail[7] since we will use the Allen quantity index concept to distinguish VW's concept of a true quantity index from true quantity indexes that have been defined in the literature.

A fourth and somewhat different concept for a true index is associated with Afriat (1981) and Dowrick and Quiggin (1997). If for each bilateral comparison subsumed within a broader multilateral comparison, the maximum of all the chained Paasche paths between the two periods or regions is less than the minimum of all the chained Laspeyres paths, then any index that for all pairs of bilateral comparisons lies between these so-called Afriat bounds is defined as true. The resulting index is true in the sense that there exists a nonparametric utility function that rationalizes the data and generates Konüs indexes that are identically equal to it. In our opinion, however, this alternative usage of the word "true" is misleading because it is at odds with a large literature that uses this term differently. VW seem to have been influenced by this fourth concept.

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[4] Samuelson (1974) called this a money metric measure of utility.

[5] See Diewert (1981) for references to the literature.

[6] Shephard (1953) was an early pioneer in developing this theory of aggregation.

[7] See Diewert (1981) and Caves, Christensen and Diewert (1982) for additional material on this index concept.

Note that the concepts of a Konüs true price index and an Allen true quantity index are not immediately "practical" concepts since they assume that the functional form for the consumer's utility function (or its dual cost function) is known.[8] Note also that definition (2) for a true Konüs price index is defined for any given utility function U satisfying the regularity conditions listed above (with dual cost function C) for all strictly positive price vectors $\mathbf{p}^1$ and $\mathbf{p}^2$ and for all strictly positive reference quantity vectors $\mathbf{q}$. Similarly, definition (4) for a true Allen quantity index is defined for any given utility function U satisfying the regularity conditions listed above (again with dual cost function C), for all strictly positive quantity vectors $\mathbf{q}^1$ and $\mathbf{q}^2$ and for all strictly positive reference price vectors $\mathbf{p}$.

3. The VW System of True Quantity Indexes

Having reviewed the literature on bilateral true indexes, we are now ready to consider van Veelen and van der Weide's (VW's) (2008) multilateral concepts for a system of true quantity indexes. They assume that price and quantity data, $\mathbf{p}^m$ and $\mathbf{q}^m$ for $m = 1,\ldots,M$ are available for say M countries. Denote the N by M matrix of country price data by $\mathbf{P} \equiv [\mathbf{p}^1,\mathbf{p}^2,\ldots,\mathbf{p}^M]$ and the N by M matrix of country quantity data by $\mathbf{Q} \equiv [\mathbf{q}^1,\mathbf{q}^2,\ldots,\mathbf{q}^M]$. A *system of VW multilateral quantity indexes* is a set of M functions, $[F_1(\mathbf{P},\mathbf{Q}), F_2(\mathbf{P},\mathbf{Q}),\ldots,F_M(\mathbf{P},\mathbf{Q})] \equiv F(\mathbf{P},\mathbf{Q})$ where F is a vector valued function whose components are the country relative quantity aggregates, the $F_m(\mathbf{P},\mathbf{Q})$.

VW (2008; 1724-1725) provide three alternative definitions for the concept of a true quantity index in the multilateral context. These definitions are of interest, but none of their definitions coincide with the definitions for a true index that already exist in the literature. Their third definition of a true multilateral system is closest to what we think is the definition in the literature on true indexes and so we will repeat it here:

VW's Third Definition: The vector valued function $F(\mathbf{P},\mathbf{Q})$ is a *true system of multilateral quantity indexes* for the utility function U if for all data sets $(\mathbf{P},\mathbf{Q})$ that U rationalizes, the following inequalities hold:

$$F_j(\mathbf{P},\mathbf{Q}) > F_k(\mathbf{P},\mathbf{Q}) \leftrightarrow U(\mathbf{q}^j) > U(\mathbf{q}^k) \quad \text{for all } 1 \le j,k \le M. \tag{6}$$

[8] However, if preferences have been estimated econometrically, then these true index number concepts do become "practical". Moreover, one can construct observable nonparametric bounds to these indexes and under certain conditions, these bounds again become practical; see Pollak (1983) and Diewert (1981) for expositions of this bounds approach to true indexes. The working paper version of Pollak (1983) was issued in (1971).

4. An Allen True Multilateral System of Quantity Indexes

Now we consider alternative definitions for a true multilateral system of quantity indexes based on the existing literature on true indexes. In the case where preferences are nonhomothetic, the *system of true Allen multilateral quantity indexes* consists of the following M functions where the positive price vector $\mathbf{p}$ is an arbitrarily chosen reference price vector:

$$(7) \qquad C(U(\mathbf{q}^1),\mathbf{p}), C(U(\mathbf{q}^2),\mathbf{p}), \ldots, C(U(\mathbf{q}^M),\mathbf{p})$$

where as usual, C is the cost or expenditure function that is dual to the utility function U. In the case where preferences are linearly homogeneous, then it is not necessary to specify a reference price vector and the *system of true multilateral quantity indexes* in this case becomes just the vector of country utilities:

$$(8) \qquad U(\mathbf{q}^1), U(\mathbf{q}^2), \ldots, U(\mathbf{q}^M) .$$

Comparing (6), (7) and (8), it can be seen that (8) could be regarded as a special case of the VW definition; i.e., if we set $F_j(\mathbf{P},\mathbf{Q})$ equal to $U(\mathbf{q}^j)$, then it can be seen that the VW definition of a true multilateral index is equivalent to the definition of a true index that is in the traditional literature but of course, we need the homothetic preferences assumption in order to get this equivalence. In the general case where preferences are not homothetic, then it can be seen that the "traditional" definition of a true set of multilateral indexes (7) cannot be put into the VW form (6). Using the VW definition of a true system, their functions F_j depend on two matrices of observed price and quantity data, $\mathbf{P}$ and $\mathbf{Q}$. In contrast, using the Allen definition of a true system, the counterpart functions to the F_j depend only on the observed country j quantity vector $\mathbf{q}^j$ and the reference price vector $\mathbf{p}$. Thus, the definition that VW suggest differs from the literature's existing definition of a true index.[9]

5. Traditional Definitions for Exact Indexes

We now turn our attention to the concept of an exact index as it exists in the index number literature. We will first look at the concept of an exact index in the bilateral context; i.e., where we are comparing only two price quantity situations.

The concept of an exact index number formula dates back to the pioneering contributions of Konüs and Byushgens (1926) in the context of bilateral index number theory.[10] In the price index context, the theory starts with a given bilateral index number formula for an axiomatic price index P which is a function of the price and quantity vectors pertaining to two situations

[9] Of course, VW are entitled to make whatever definitions they find convenient. Our point is that they should carefully note that they are changing a well established definition of a true index.

[10] For additional material on the contributions of Konüs and Byushgens, see Afriat (1972) and Diewert (1976).

(time periods or countries) where the prices are positive, say $P(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2)$. The function P is supposed to reflect the price level in, say, country 2 relative to the price level in country 1.

Now assume that the data $\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2$ pertaining to the two countries is generated by utility maximizing behavior on the part of an economic agent, where the utility function $U(\mathbf{q})$ is defined over the nonnegative orthant, and is nonnegative, linearly homogeneous, increasing (if all components of $\mathbf{q}$ increase) and concave. The unit cost function that is dual to $U(\mathbf{q})$ is $c(\mathbf{q})$.

The existing literature defines $P(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2)$ to be an *exact price index* for $U(\mathbf{q})$ and its dual unit cost function $c(\mathbf{p})$ if

(9) $\quad P(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2) = c(\mathbf{p}^2)/c(\mathbf{p}^1)$.

The equality (9) is supposed to hold for all strictly positive price vectors $\mathbf{p}^1$ and $\mathbf{p}^2$ (and, of course, the corresponding $\mathbf{q}^1$ and $\mathbf{q}^2$ are assumed to be solutions to the cost minimization problems defined by (1).

There is an analogous theory for exact quantity indexes, $Q(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2)$. Under the homothetic (actually linearly homogeneous) preference assumptions made in the previous paragraph and under the assumption that the data are consistent with cost minimizing behavior (1), the existing literature says that $Q(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2)$ is an *exact quantity index* for $U(\mathbf{q})$ if

(10) $\quad Q(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2) = U(\mathbf{q}^2)/U(\mathbf{q}^1)$.

Many examples of exact bilateral price and quantity indexes are presented in Konüs and Byushgens (1926), Afriat (1972), Pollak (1983) and Diewert (1976).

Note that the above theory of exact quantity indexes does not guarantee that a given set of bilateral price and quantity vectors, $\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2$, are actually consistent with utility maximizing (or cost minimizing) behavior. The theory only says that given a particular functional form for U, given arbitrary strictly positive price vectors $\mathbf{p}^1$ and $\mathbf{p}^2$, and given that $\mathbf{q}^i$ solves the cost minimization problem (1) for $i = 1,2$, then a given function of 4N variables $Q(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2)$ is an exact quantity index for the preferences defined by U if (10) holds. The problem that VW have uncovered with this definition has to do with the assumption that (10) holds for *all* strictly positive price vectors $\mathbf{p}^1$ and $\mathbf{p}^2$: this is not always the case for many of the commonly used exact index number formulae. We will return to this important point later.

The theory of exact quantity indexes in the multilateral situation is not as well developed as in the bilateral context. Note that in the bilateral context, an exact index number formula is exact for a utility ratio; i.e., the exact index number literature does not attempt to determine utility up to a cardinal scale but rather it only attempts to determine the utility ratio between the two situations. In the multilateral context, we could attempt to determine utility ratios relative to a numeraire country but then one country would be asymmetrically singled out to play the role of the numeraire country. Thus Diewert (1988) developed an axiomatic approach to multilateral quantity indexes that is based on a system of *country share functions*,

$[S_1(\mathbf{P},\mathbf{Q}),S_2(\mathbf{P},\mathbf{Q}),\ldots,S_M(\mathbf{P},\mathbf{Q})] \equiv S(\mathbf{P},\mathbf{Q})$ where S is a vector valued function whose components are the country relative quantity aggregates, the $S_m(\mathbf{P},\mathbf{Q})$, where each S_m represents the share of country m in world output (or consumption).[11] For all practical purposes, Diewert's system of share functions, $S(\mathbf{P},\mathbf{Q})$, is equivalent to VW's system of multilateral indexes, $F(\mathbf{P},\mathbf{Q})$.

Diewert (1999; 20-23) developed a theory of exact indexes in the multilateral context and we will explain his theory below.[12]

The basic assumption in Diewert's economic approach to multilateral indexes is that the country m quantity vector $\mathbf{q}^m$ is a solution to the following country m utility maximization problem:

(11) $\max_{\mathbf{q}}\{U(\mathbf{q}) : \mathbf{p}^m\mathbf{q} = \mathbf{p}^m\mathbf{q}^m\} = u^m,$

for $m = 1,\ldots,M$ where $u^m \equiv U(\mathbf{q}^m)$ is the utility level for country m, $\mathbf{p}^m$ is the vector of strictly positive prices for outputs that prevail in country m for $m = 1,\ldots,M$, and U is a linearly homogeneous, increasing and concave utility function that is assumed to be the same across countries.[13] As usual, the utility function has a *dual unit cost or expenditure function* $c(\mathbf{p})$ which is defined as the minimum cost or expenditure required to achieve a unit utility level if the consumer faces the positive commodity price vector $\mathbf{p}$.[14] Since consumers in country m are assumed to face the positive prices $\mathbf{p}^m$, we have the following equalities:

(12) $c(\mathbf{p}^m) \equiv \min_{\mathbf{q}}\{\mathbf{p}^m\mathbf{q} : U(\mathbf{q}) \geq 1\} \equiv P^m; \qquad m = 1,\ldots,M,$

where P^m is the (unobserved) minimum expenditure that is required for country m to achieve a unit utility level when it faces its prices $\mathbf{p}^m$, which can also be interpreted as country m's PPP, or Purchasing Power Parity. Under the above assumptions, it can be shown that the country data satisfy the following equations:

(13) $\mathbf{p}^m\mathbf{q}^m = c(\mathbf{p}^m)U(\mathbf{q}^m) = P^m u^m; \qquad m = 1,\ldots,M.$

In order to make further progress, we assume that the unit cost function $c(\mathbf{p})$ is once continuously differentiable with respect to the components of $\mathbf{p}$. Then Shephard's Lemma implies the following equations which relate the country m quantity vectors $\mathbf{q}^m$ to the country m price vectors $\mathbf{p}^m$ and utility levels u^m:

(14) $\mathbf{q}^m = \nabla c(\mathbf{p}^m)u^m; \qquad m = 1,\ldots,M.$

[11] This multilateral axiomatic approach was further refined by Balk (1996) and Diewert (1999).

[12] See also Diewert (2008).

[13] Note that in Diewert's multilateral approach to exact indexes (1999) (2008), he did not consider the case of nonhomothetic preferences whereas in the bilateral case, Diewert (1976) did consider the nonhomothetic case.

[14] The unit cost function $c(\mathbf{p})$ is an increasing, linearly homogeneous and concave function in p for $p \gg 0_N$.

Now we are ready to define the concept of exactness for a multilateral share system. We say that *the multilateral system of share functions,* $S(\mathbf{P},\mathbf{Q})$, *is exact* for the linearly homogeneous utility function U and its differentiable dual unit cost function c if the following system of equations is satisfied for all strictly positive country price vectors $\mathbf{P} \equiv [\mathbf{p}^1,\ldots,\mathbf{p}^M]$ and all positive utility levels $u^1,\ldots,u^M$:

$$(15)\qquad \frac{S_i(P,\nabla c(\mathbf{p}^1)u^1,\nabla c(\mathbf{p}^2)u^2,\ldots,\nabla c(\mathbf{p}^M)u^M}{S_j(P,\nabla c(\mathbf{p}^1)u^1,\nabla c(\mathbf{p}^2)u^2,\ldots,\nabla c(\mathbf{p}^M)u^M} = \frac{u^i}{u^j}; \qquad i,j=1,\ldots,M.$$

Thus an exact multilateral share system gives us exactly the underlying utilities up to an arbitrary positive scaling factor. Diewert (1999, 2008) gives many examples of exact multilateral systems. Diewert also goes on to define a *superlative multilateral system* to be an exact system where the underlying utility function U or dual unit cost function can approximate an arbitrary linearly homogeneous function to the second order around any given data point.

As in the bilateral case, VW have uncovered a problem with our definition (15) above for an exact multilateral system. The problem is that it is assumed that (15) holds for *all* strictly positive price vector matrices $\mathbf{P}$: this is not always the case for many of the commonly used exact index number formulae. We will return to this important point in the following section.

Van Veelen and van der Weide (2008; 1723) also give their definition of an exact multilateral system (which we will not reproduce here due to its complexity). However, their definition is rather far from the above definition of multilateral exactness that is out there in the literature.[15]

In our view, the "problem" with the VW definitions of true and exact indexes is that they are mixing up these theoretical concepts (as they exist in the index number literature) with a related but different question: namely, is a given set of, say, M price and quantity vectors consistent with utility maximizing behavior under various assumptions? This latter question is an interesting one and there is certainly room for more research in this area. However, some care should be taken to not redefine well established concepts as this research takes place.

6. The Problems Associated with Finding the Regularity Region for Exact Indexes

In the previous section, we noted that there can be a problem with some well known exact index number formulae in that the exactness property does not always hold for *all* strictly positive prices. We will explain the problem by giving two examples of exact index number formulae: one where there is no problem, and a second where there could be a problem.

[15] A major problem with their definition is this: the VW definition is conditional on a set of admissible price and quantity vectors D but this admissible set is not well specified. If we take the set D to be a single price quantity point for each country where the country price vectors are all equal to the same **p** and the country quantity vectors are all equal to the same **q**, and the function F treated countries in a symmetric manner, then F would be exact for any utility function.

Example 1: The Jevons Price Index

Suppose each consumer's unit cost function c has the following Cobb-Douglas functional form:[16]

$$(16) \qquad c(\mathbf{p}) \equiv \beta \prod_{n=1}^{N} p_n^{\alpha_n} ,$$

where the α_n are positive constants which sum to one and β is a positive constant. If we are comparing the level of prices in country 2 relative to country 1, then the Jevons (1865) price index, P_J is defined as the first line in (17):

$$(17) \qquad \begin{aligned} P_J(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2) &\equiv \prod_{n=1}^{N} (p_n^2 / p_n^1)^{s_n^1} \\ &= c(\mathbf{p}^2)/c(\mathbf{p}^1), \end{aligned}$$

where the unit cost function c is defined by (16) and the nth expenditure share for country 1, s_n^1, is defined as $p_n^1 q_n^1 / \mathbf{p}^1\mathbf{q}^1$ for $n = 1,\ldots,N$. Thus under the assumption that consumers in the two countries have identical Cobb Douglas preferences $U(\mathbf{q})$ that are dual to the unit cost function c defined by (16) and assuming cost minimizing behavior on the part of consumers in both countries, then (17) tells us that the true Konüs price index between the two countries is *exactly* equal to the observable Jevons price index $P_J(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2)$ and that *this equality will hold for all strictly positive price vectors* $\mathbf{p}^1$ and $\mathbf{p}^2$ for the two countries. The corresponding Jevons quantity index $Q_J(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2)$ is defined as the expenditure ratio divided by the Jevons price index and we have the following equalities:

$$(18) \qquad \begin{aligned} Q_J(p^1,p^2,q^1,q^2) &\equiv p^2q^2/p^1q^1 P_J(p^1,p^2,q^1,q^2) \\ &= U(q^2)/U(q^1). \end{aligned}$$

Thus under our assumptions on consumer behavior, (18) tells us that the true Allen quantity index between the two countries is *exactly* equal to the observable Jevons quantity index $Q_J(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2)$ and again, this equality will hold for all strictly positive price vectors $\mathbf{p}^1$ and $\mathbf{p}^2$ for the two countries (with the corresponding quantity vectors $\mathbf{q}^1$ and $\mathbf{q}^2$ being endogenously determined). If we want to put the above results into the format that VW use, then the VW system of country quantity indexes could be defined as follows:

$$(19) \qquad F_1(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2) \equiv 1;\; F_2(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2) \equiv Q_J(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2).$$

Thus the theory of exact indexes works well under the assumption of Cobb Douglas preferences. However, note that the theory does not investigate whether consumers in the two countries actually minimize their costs of achieving their utility targets and whether they actually

[16] The Cobb Douglas case is treated in some detail by Afriat (1972) and Pollak (1983).

have Cobb Douglas preferences.[17] The theory is a conditional one: if consumers have certain preferences and if they engage in cost minimizing behavior, then their true price (or quantity) index will be exactly equal to a certain index number formula which in turn is a function of the observable price and quantity data pertaining to the two countries.

We turn to our second example of an exact index.

Example 2: The Fisher Price Index

Suppose each consumer has preferences that are dual to the following unit cost function:[18]

(20) $\quad c(\mathbf{p}) \equiv (\mathbf{p}^T\mathbf{B}\mathbf{p})^{1/2}\,;\ \mathbf{B} = \mathbf{B}^T,$

where $\mathbf{B}$ is an N by N symmetric matrix which has one positive eigenvalue (with a strictly positive eigenvector) and the remaining N–1 eigenvalues are negative or zero. The vector of first order partial derivatives of this unit cost function, $\nabla c(\mathbf{p})$, and the matrix of second order partials, $\nabla^2 c(\mathbf{p})$, are equal to the following expressions:

(21) $\quad \nabla c(\mathbf{p}) = \mathbf{B}\mathbf{p}/(\mathbf{p}^T\mathbf{B}\mathbf{p})^{1/2}\,;$

(22) $\quad \nabla^2 c(\mathbf{p}) = (\mathbf{p}^T\mathbf{B}\mathbf{p})^{-1/2}\{\mathbf{B} - \mathbf{B}\mathbf{p}(\mathbf{p}^T\mathbf{B}\mathbf{p})^{-1}\mathbf{p}^T\mathbf{B}\}\,.$

At this point, we encounter the problem which we believe bothered VW; namely, that the unit cost function defined by (20) will generally not provide a representation of well behaved consumer preferences for all strictly positive price vectors $\mathbf{p}$. In order for a unit cost function to provide a valid global representation of homothetic preferences, it must be a nondecreasing, linearly homogeneous and concave function over the positive orthant. However, in order for c to provide a valid local representation of preferences, we need only require that $c(\mathbf{p})$ be positive, nondecreasing, linearly homogeneous and concave over a convex subset of prices, say S, where S has a nonempty interior.[19] It is obvious that $c(\mathbf{p})$ defined by (20) is linearly homogeneous. The nondecreasing property will hold over S if the gradient vector $\nabla c(\mathbf{p})$ defined by (21) is strictly positive for $p \in S$ and the concavity property will hold if $\nabla^2 c(\mathbf{p})$ defined by (22) is a negative semidefinite matrix for $p \in S$. We will show how the regularity region S can be determined shortly but first, we will indicate why the $c(\mathbf{p})$ defined by (20) is a flexible functional form[20] since this explanation will help us to define an appropriate region of regularity.

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[17] An implication of the Cobb Douglas preferences model is that the expenditure shares in the two countries should be equal; i.e., we should have $s_n^1 = s_n^2$ for $n = 1,\cdots,N$. Of course, in the real world, these restrictions are unlikely to be satisfied.

[18] This is a special case of a functional form due to Denny (1974), which Diewert (1976; 131) called the quadratic mean of order r unit cost function, with $r = 2$.

[19] See Blackorby and Diewert (1979) for more details on local representations of preferences using duality theory.

[20] A flexible functional form is one that is capable of providing a second order approximation to an arbitrary function in the class of functions under consideration; see Diewert (1976; 115) who introduced the term into the economics literature.

Let $\mathbf{p}^* >> 0_N$ be a strictly positive reference price vector and suppose that we are given an arbitrary unit cost function $c^*(\mathbf{p})$ that is twice continuously differentiable in a neighborhood around $\mathbf{p}^*$.[21] Let $\mathbf{q}^* \equiv \nabla c^*(\mathbf{p}) >> \mathbf{0}_N$ be the strictly positive vector of first order partial derivatives of $c^*(\mathbf{p}^*)$ and let $\mathbf{\Sigma} \equiv \nabla^2 c^*(\mathbf{p}^*)$ be the negative semidefinite symmetric matrix of second order partial derivatives of c^* evaluated at $\mathbf{p}^*$. Euler's Theorem on homogeneous functions implies that Σ satisfies the following matrix equation:

(23) $\quad \mathbf{\Sigma p}^* = \mathbf{0}_N$.

In order to establish the flexibility of the c defined by (20), we need only show that there are enough free parameters in the $\mathbf{B}$ matrix so that the following equations are satisfied:

(24) $\quad \nabla c(\mathbf{p}^*) = \mathbf{q}^*$;

(25) $\quad \nabla^2 c(\mathbf{p}^*) = \mathbf{\Sigma}$

In order to prove the flexibility of c defined by (20), it is convenient to reparameterize the **B** matrix. Thus we now set **B** equal to:

(26) $\quad \mathbf{B} = \mathbf{bb}^T + \mathbf{A}$,

where $\mathbf{b} >> \mathbf{0}_N$ is a positive vector and $\mathbf{A}$ is a negative semidefinite matrix which has rank equal to at most $N-1$ and it satisfies the following restrictions:

(27) $\quad \mathbf{Ap}^* = \mathbf{0}_N$.

Note that $\mathbf{bb}^T$ in (26) is a rank one positive semidefinite matrix with $\mathbf{p}^{*T}\mathbf{bb}^T\mathbf{p}^* = (\mathbf{b}^T\mathbf{p}^*)^2 > 0$ and $\mathbf{A}$ is a negative semidefinite matrix and satisfies $\mathbf{p}^{*T}\mathbf{Ap}^* = \mathbf{0}$. Thus it can be seen that $\mathbf{B}$ is a matrix with one positive eigenvalue and the other eigenvalues are negative or zero.

Substitute (21) into (24) in order to obtain the following equation:

(28) $\quad \mathbf{q}^* = \mathbf{Bp}^* / (\mathbf{p}^{*T}\mathbf{Bp}^*)^{-1/2}$

$$= [\mathbf{bb}^T + \mathbf{A}]\mathbf{p}^* / (\mathbf{p}^{*T}[\mathbf{bb}^T + \mathbf{A}]\mathbf{p}^*)^{-1/2} \qquad \text{using (26)}$$

[21] Of course, in addition, we assume that c^* satisfies the appropriate regularity conditions for a unit cost function. Using Euler's Theorem on homogeneous functions, the fact that c^* is linearly homogeneous and differentiable at $\mathbf{p}^*$ means that the derivatives of c^* satisfy the following restrictions: $c^*(\mathbf{p}^*) = p^{*T}\nabla c^*(\mathbf{p}^*)$ and $\nabla^2 c^*(\mathbf{p}^*)\mathbf{p}^* = \mathbf{0}_N$. The unit cost function c defined by (20) satisfies analogous restrictions at $\mathbf{p} = \mathbf{p}^*$. These restrictions simplify the proof of the flexibility of c at the point $\mathbf{p}^*$.

$$= \mathbf{b}\mathbf{b}^{T}\mathbf{p}^{*}/(\mathbf{p}^{*T}\mathbf{b}\mathbf{b}^{T}\mathbf{p}^{*})^{1/2} \qquad \text{using (27)}$$

$$= \mathbf{b}.$$

Thus if we choose **b** equal to $\mathbf{q}^{*}$, equation (24) will be satisfied. Now substitute (22) into (23) and obtain the following equation:

$$(29) \qquad \mathbf{\Sigma} = (\mathbf{p}^{*T}\mathbf{B}\mathbf{p}^{*})^{-1/2}\{\mathbf{B} - \mathbf{B}\mathbf{p}^{*}(\mathbf{p}^{*T}\mathbf{B}\mathbf{p}^{*})^{-1}\mathbf{p}^{*T}\mathbf{B}\}$$

$$= (\mathbf{p}^{*T}\mathbf{b}\mathbf{b}^{T}\mathbf{p}^{*})^{-1/2}\{\mathbf{b}\mathbf{b}^{T} + \mathbf{A} - \mathbf{b}\mathbf{b}^{T}\mathbf{p}^{*}(\mathbf{p}^{*T}\mathbf{b}\mathbf{b}^{T}\mathbf{p}^{*})^{-1}\mathbf{p}^{*T}\mathbf{b}\mathbf{b}^{T} \qquad \text{using (26) and (27)}$$

$$= (\mathbf{b}^{T}\mathbf{p}^{*})^{-1}\mathbf{A} \qquad \text{using } \mathrm{b}^{T}\mathrm{p}^{*} > 0.$$

Thus if we choose **A** equal to $(\mathbf{b}^{T}\mathbf{p}^{*})\mathbf{\Sigma}$, equation (25) will be satisfied and the flexibility of c defined by (20) is established.[22]

Now we can define the region of regularity for c defined by (20).[23] Consider the following set of prices:

$$(30) \qquad S \equiv \{\mathbf{p} : \mathbf{p} >> \mathbf{0}_N ; \mathbf{B}\mathbf{p} >> 0_N\}.$$

If $\mathbf{p} \in S$, then it can be seen that $c(\mathbf{p}) = (\mathbf{p}^{T}\mathbf{B}\mathbf{p})^{1/2} > 0$ and using (21), $\nabla c(\mathbf{p}) >> \mathbf{0}_N$. However, it is much more difficult to establish the concavity of c(**p**) over the set S. We first consider the case where the matrix **B** has full rank so that it has one positive eigenvalue and $N-1$ negative eigenvalues. Let $\mathbf{p} \in S$ and using equation (22), we see that $\nabla^{2}c(\mathbf{p})$ will be negative semidefinite if and only if the matrix $\mathbf{M}$ defined as:

$$(31) \qquad \mathbf{M} \equiv \mathbf{B} - \mathbf{B}\mathbf{p}(\mathbf{p}^{T}\mathbf{B}\mathbf{p})^{-1}\mathbf{p}^{T}\mathbf{B}$$

is negative semidefinite. Note that **M** is equal to the matrix **B** plus the rank 1 negative semidefinite matrix $-\ \mathbf{B}\mathbf{p}(\mathbf{p}^{T}\mathbf{B}\mathbf{p})^{-1}\mathbf{p}^{T}\mathbf{B}$. **B** has one positive eigenvalue and the remaining eigenvalues are 0 or negative. Since **M** is **B** plus a negative semidefinite matrix, the eigenvalues of $\mathbf{M}$ cannot be greater than the eigenvalues of $\mathbf{B}$. Now consider two cases; the first case where $\mathbf{B}$ has one positive and $N-1$ negative eigenvalues and the second case where $\mathbf{B}$ has $N-1$ negative or zero eigenvalues in addition to its positive eigenvalue. Consider case 1, let $\mathbf{p} \in S$ and calculate $\mathbf{M}\mathbf{p}$:

$$(32) \qquad \mathbf{M}\mathbf{p} = [\mathbf{B} - \mathbf{B}\mathbf{p}(\mathbf{p}^{T}\mathbf{B}\mathbf{p})^{-1}\mathbf{p}^{T}\mathbf{B}]\mathbf{p} = \mathbf{0}_N.$$

The above equation shows that $\mathbf{p} \neq \mathbf{0}_N$ is an eigenvector of **M** that corresponds to a 0 eigenvalue. Now the addition of a negative semidefinite matrix to **B** can only make the $N-1$ negative eigenvalues of **B** more negative (or leave them unchanged) so we conclude that the

[22] We need to check that **A** is negative semidefinite (which it is since it is a positive multiple of the negative semidefinite substitution matrix $\mathbf{\Sigma}$) and that **A** satisfies the restrictions in (27), since we used these restrictions to derive (28) and the second line in (29). But **A** does satisfy (27) since $\mathbf{\Sigma}$ satisfies (23).

[23] The region of regularity can sometimes be extended to the closure of the set S.

addition of the negative semidefinite matrix $-\mathbf{Bp}(\mathbf{p}^T\mathbf{Bp})^{-1}\mathbf{p}^T\mathbf{B}$ to $\mathbf{B}$ has converted the positive eigenvalue of $\mathbf{B}$ into a zero eigenvalue and hence $\mathbf{M}$ is negative semidefinite. Case 2 follows using a perturbation argument.

We are now in a position to exhibit an index number formula that is consistent with the preferences that are dual to c defined by (20). Thus we again consider the two country case and define the Fisher (1922) ideal price index P_F as follows:

(33) $$F_F(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2) \equiv [\mathbf{p}^2\mathbf{q}^1\mathbf{p}^2\mathbf{q}^2/\mathbf{p}^1\mathbf{q}^1\mathbf{p}^1\mathbf{q}^2]^{1/2}.$$

Assume that $c(\mathbf{p})$ is defined by (20) and S defined by (30) is nonempty. Suppose that consumers in the two countries have preferences $U(\mathbf{q})$ that are locally dual to $c(\mathbf{p})$ and that the country price vectors, $\mathbf{p}^1$ and $\mathbf{p}^2$, both belong to S. Finally, assume that consumers in both countries engage in cost minimizing behavior. Then, under all these hypotheses, we have the following equality:[24]

(34) $$F_F(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2) = c(\mathbf{p}^2)/c(\mathbf{p}^1).$$

Thus under our hypotheses, (34) tells us that the true Konüs price index between the two countries is *exactly* equal to the observable Fisher price index $P_F(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2)$ and that *this equality will hold for all strictly positive price vectors* $\mathbf{p}^1$ and $\mathbf{p}^2$ for the two countries that belong to the set S. As was the case for the Jevons index, the corresponding Fisher quantity index $Q_F(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2)$ can be defined as the expenditure ratio divided by the Fisher price index and we have the following equalities:

(35) $$\begin{aligned} Q_F(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2) &\equiv \mathbf{p}^2\mathbf{q}^2/\mathbf{p}^1\mathbf{q}^1 P_F(\mathbf{p}^1,\mathbf{p}^2,\mathbf{q}^1,\mathbf{q}^2) \\ &= U(\mathbf{q}^2)/U(\mathbf{q}^1), \end{aligned}$$

where U is the utility function that is locally dual to c.

What are we to make of the above results in the light of the criticisms of VW? We think that VW are justified in noting the limitations of the above theory of exact index numbers. Some of these limitations are:

- All consumers in all countries in the comparison are generally assumed to have the same homothetic preferences;
- There are no checks done on the data to see if consumers really are maximizing a common linearly homogeneous utility function and finally,
- The exact result (for example (34)) may not hold for all positive price vectors pertaining to the countries in the comparison but may only hold for a subset $\mathbf{S}$ of prices and it will usually be difficult to figure out exactly what this set is.

[24] See Diewert (1976; 134) and specialize his result to the case where $r=2$.

Our response to these valid criticisms is the following one. We regard exact superlative indexes (indexes which are exact for flexible functional forms) as a useful screening device. There are an infinite number of index number formulae out there and it is useful to distinguish formulae that have "good" economic properties under at least some conditions.[25] However, it is always useful to consider other noneconomic approaches to index number theory and it is perhaps "ideal" if the different approaches lead to the same index number formulae. Thus North American price statisticians tend to favor the use of the Fisher or Törnqvist Theil (1967) bilateral formula because of the connection of these indexes with the economic approach to index number theory whereas European statisticians tend to favor the axiomatic approach or the stochastic approach[26] to index number theory. However, strong axiomatic justifications for the use of the Fisher index can be given[27] and a strong axiomatic for the Törnqvist Theil formula can also be given.[28] Furthermore, the Törnqvist Theil formula also does well from the viewpoint of the stochastic approach. Thus at the current state of index number theory, it appears that the Fisher and the Törnqvist Theil indexes are pretty good choices from multiple points of view.[29]

7. The Distinction Between the Axiomatic and Economic Approaches

Although VW make many good points in their note, they make some points which we find are problematical. Consider the following quotation:

> "In the literature, two approaches to index numbers are distinguished: the axiomatic approach and the economic approach. ... In Neary's paper the difference is described as one between an approach that does and an approach that does not assume that quantities arise from optimizing behavior. ... We will argue that a more accurate description is that the difference lies in whether or not optimizing agents, or representative consumers, are assumed to optimize the *same* utility function." Matthijs van Veelen and Roy van der Weide (2008; 1722).

We do not agree with the above assertions: it seems to us that the economic approach definitely takes prices as exogenous variables and treats quantities as being endogenous, whereas the axiomatic approach treats both prices and quantities as being exogenous. That is, we agree with the consensus view, as stated in Neary (2004) and Balk (2008), which can be traced back at least to Frisch (1936). We do not think it is particularly helpful to try and blend the two approaches (although in the end, they may lead to the same index number formulae).

VW argue that an advantage of the axiomatic approach is that it allows for heterogeneity in preferences. We take issue with this claim. The economic approach allows for heterogeneity

[25] There are even an infinite number of superlative formulae as indicated by Diewert (1976) but Hill (2006) noted that not all of these formulae are really that super.

[26] See Theil (1967), Selvanathan and Rao (1994) and Clements, Izan and Selvanathan (2006) on the stochastic approach to index numbers.

[27] See Diewert (1992) and Balk (1995).

[28] See Diewert (2004).

[29] This argument follows along similar arguments made by Diewert (1997). Also Diewert (1978) showed that the Fisher and Törnqvist Theil indexes will numerically approximate each other to the second order around an equal price and quantity point. Thus, in the time series context, it will often not matter which of these indexes is used.

too, across households in each period and in tastes across periods. Pollak (1980, 1981, 1983) and Diewert (1984, 2001) extend the Konüs true index to the case of heterogeneous agents. For example, a plutocratic Konüs true index is defined as follows:

$$(36) \qquad P_K(\mathbf{p}^1, \mathbf{p}^2, \mathbf{q}_1, \ldots, \mathbf{q}_H) \equiv \sum_{h=1}^{H} C_h[U_h(\mathbf{q}_h), \mathbf{p}^2] / \sum_{h=1}^{H} C_h[U_h(\mathbf{q}_h), \mathbf{p}^1],$$

where h indexes the households.[30] A plutocratic Konüs true index measures the change in the minimum cost of each household h achieving its reference utility level $U_h(\mathbf{q}_h)$ from period 1 to period 2. The plutocratic Konüs true index as formulated in (36) therefore explicitly allows preferences to differ across households. Similarly, true indexes that allow preferences to change over time are derived by Caves, Christensen and Diewert (1982) and Balk (1989). In short, the economic approach is more flexible than VW's analysis suggests.

8. Conclusion

Van Veelen and van der Weide (2008) have raised a number of contentious issues that deserve closer scrutiny. While we take issue with some of their findings, we commend them for providing a fresh perspective on an old topic.

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[30]Pollak (1989) and Diewert (2001) also consider generalizations that allow for environmental variables. An index that allows for changes in environmental variables is sometimes referred to as unconditional, while one that does not is referred to as conditional.

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