

Chapter 7

MEASURING PRODUCTIVITY CHANGE WITHOUT NEOCLASSICAL ASSUMPTIONS: A CONCEPTUAL ANALYSIS

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1. Introduction

The measurement of productivity change (or difference) is usually based on models that make use of strong assumptions such as competitive behaviour and constant returns to scale. This survey discusses the basics of productivity measurement and shows that one can dispense with most if not all the usual, neoclassical assumptions. By virtue of its structural features, the measurement model is applicable to individual establishments and aggregates such as industries, sectors, or economies.

The methodological backing of productivity measurement and growth accounting usually goes like this. The (aggregate) production unit considered has an input side and an output side, and there is a production function that links output quantities to input quantities. This production function includes a time variable, and the partial derivative of the production function with respect to the time variable is called technological change (or, in some traditions, multi- or total factor productivity change). Further, it is assumed that the production unit acts in a competitive environment; that is, input and output prices are assumed as given. Next, it is assumed that the production unit acts in a profit maximizing manner (or, it is said to be ‘in equilibrium’), and that the production function exhibits constant returns to scale. Under these assumptions it then appears that output quantity growth (defined as the output-share-weighted mean of the individual output quantity growth rates) is equal to input quantity growth (defined as the input-share-weighted mean of the individual input quantity growth rates) plus the rate of technological change (or, multi- or total factor productivity growth).

For the empirical implementation one then turns to National Accounts, census and/or survey data, in the form of nominal values and deflators (price indices). Of course, one cannot avoid dirty hands by making various imputations where direct observations failed or were impossible (as in the case of labour input of self-employed workers). In the case of capital inputs the prices, necessary for the computation of input shares, cannot be observed, but must be computed as unit user costs. The single degree of freedom that is here available, namely the rate

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of return, is used to ensure that the restriction implied by the assumption of constant returns to scale, namely that profit equals zero, is satisfied. This procedure is usually rationalized by the assumption of perfect foresight, which in this case means that the ex post calculated capital input prices can be assumed as ex ante given to the production unit, so that they can be considered as exogenous data for the unit's profit maximization problem.

This account is, of course, somewhat stylized, since there occur many, smaller or larger, variations on this theme in the literature. Recurring, however, are a number of so-called neo-classical assumptions: 1) a technology that exhibits constant returns to scale, 2) competitive input and output markets, 3) optimizing behaviour, and 4) perfect foresight. A fine example from academia is provided by Jorgenson, Ho and Stiroh (2005, pp. 23, 37), while the Sources and Methods publication of Statistics New Zealand (2006) shows that the neo-classical model has also deeply invaded official statistical agencies.² An interesting position is taken by the EU KLEMS Growth and Productivity Accounts project. Though in their main text Timmer *et al.* (2007) adhere to the Jorgenson, Ho and Stiroh framework, there is a curious footnote saying:

“Under strict neo-classical assumptions, MFP [multifactor productivity] growth measures disembodied technological change. *In practice* [my emphasis], MFP is derived as a residual and includes a host of effects such as improvements in allocative and technical efficiency, changes in returns to scale and mark-ups as well as technological change proper. All these effects can be broadly summarized as “improvements in efficiency”, as they improve the productivity with which inputs are used in the production process. In addition, being a residual measure MFP growth also includes measurement errors and the effects from unmeasured output and inputs.”

There are more examples of authors who exhibit similar concerns, without, however, feeling the need to adapt their conceptual framework.

I believe that for an official statistical agency, whose main task it is to provide statistics to many different users for many different purposes, it is disconcerting to have such strong, and often empirically refuted, assumptions built into the methodological foundations of productivity and growth accounting statistics. This especially applies to the behavioral assumptions numbered 2, 3 and 4. There is ample evidence that, on average, markets are not precisely competitive; that producers' decisions frequently turn out to be less than optimal; and that managers almost invariably lack the magical feature of perfect foresight. Moreover, the environment in which production units operate is not so stable as the assumption of a fixed production function seems to claim.

But I also believe that it is possible, and even advisable, to avoid making such assumptions. In a sense I propose to start where the usual story ends, namely at the empirical side.³ For any production unit, the total factor productivity index is then *defined* as an output

² The neo-classical model figured already prominently in the 1979 report of the U.S. National Research Council's Panel to Review Productivity Statistics (Rees 1979). An overview of national and international practice is provided by the regularly updated *OECD Compendium of Productivity Indicators*, available at www.oecd.org/statistics/productivity.

³ There is another, minor, difference between my approach and the usual story. The usual story runs in the framework of continuous time in which periods are of infinitesimal short duration. When it then comes to implementation several approximations must be assumed. My approach does not need this kind of assumptions either, because this approach is entirely based on accounting periods of finite duration, such as years.

quantity index divided by an input quantity index. There are various options here, depending on what one sees as input and output, but the basic feature is that, given price and quantity (or value) data, this is simply a matter of index construction. There appear to be no behavioral assumptions involved, and this even applies -- as will be demonstrated -- to the construction of capital input prices. Surely, a number of imputations must be made (as in the case of the self-employed workers) and there is a fairly large number of more or less defensible assumptions involved (for instance on the depreciation rates of capital assets), but this belongs to the daily bread and butter of economic statisticians.

In my view, structural as well as behavioral assumptions enter the picture as soon as it comes to the *explanation* of productivity change. Then there are, depending on the initial level of aggregation, two main directions: 1) to explain productivity change at an aggregate level by productivity change and other factors operating at lower levels of aggregation; 2) to decompose productivity change into factors such as technological change, technical efficiency change, scale effects, input- and output-mix effects, and chance. In this case, to proceed with the analysis one cannot sidestep a technology model with certain specifications.

The contents of this paper unfold as follows. Section 2 outlines the architecture of the basic, KLEMS-Y, input-output model, with its total and partial measures of productivity change. This section also links productivity measurement and growth accounting. Section 3 proceeds with the KL-VA and K-CF models. Then it is time to discuss the measurement of capital input cost in sections 4 and 5. This gives rise to four additional input-output models, which are discussed in section 6. Section 7 is devoted to the rate of return: endogenous or exogenous, ex post or ex ante. Section 8 introduces the capital utilization rate. Section 9 considers a number of implementation issues, after which we take a look in section 10 at the Netherlands' system of productivity statistics. The conclusion can be brief.

2. The Basic Input-Output Model

Let us consider a single production unit. This could be an establishment or plant, a firm, an industry, a sector, or even an entire economy. I will simply speak of a 'unit'. For the purpose of productivity measurement, such a unit is considered as a (consolidated) input-output system. What does this mean?

For the output side as well as for the input side there is some list of commodities (according to some classification scheme). A commodity is thereby defined as a set of closely related items which, for the purpose of analysis, can be considered as "equivalent," either in the static sense of their quantities being additive or in the dynamic sense of displaying equal relative price or quantity changes. Ideally, then, for any accounting period considered (ex post), say a year, each commodity comes with a value (in monetary terms) and a price and/or a quantity. If value and price are available, then the quantity is obtained by dividing the value by the price. If value and quantity are available, then the price is obtained by dividing the value by the quantity. If both price and quantity are available, then value is defined as price times quantity. In any case, for every commodity it must be so that value equals price times quantity, the magnitudes of which of course must pertain to the same accounting period. Technically speaking, the price concept used here is the unit value. At the output side, the prices must be those received by the

unit, whereas on the input side, the prices must be those paid. Consolidation (also referred to as taking a net-sector approach) means that the unit does not deliver to itself. Put otherwise, all the intra-unit deliveries are netted out.

The situation as pictured in the preceding paragraph is typical for a unit operating on the (output) market.⁴ The question of how to deal with non-market units will be considered where appropriate.

The inputs are customarily classified according to the KLEMS format. The letter *K* denotes the class of owned, reproducible capital assets. The commodities here are the asset-types, sub-classified by age category. Cohorts of assets are assumed to be available at the beginning of the accounting period and, in deteriorated form (due to ageing, wear and tear), still available at the end of the period. Investment during the period adds entities to these cohorts, while disinvestment, breakdown, and retirement remove entities. Examples include buildings and other structures, machinery, transport and ICT equipment, and tools. As will be discussed later in detail, theory implies that the quantities sought are just the quantities of all these cohorts of assets (together representing the productive capital stock), whereas the relevant prices are their unit user costs (per type-age combination), constructed from imputed interest rates, depreciation profiles, (anticipated) revaluations, and tax rates. The sum of quantities times prices then provides the capital input cost of a production unit.⁵

The letter *L* denotes the class of labour inputs; that is, all the types of work that are important to distinguish, cross-classified for instance according to educational attainment, gender, and experience (which is usually proxied by age categories). Quantities are measured as hours worked (or paid), and prices are the corresponding wage rates per hour. Where applicable, imputations must be made for the work executed by self-employed persons. The sum of quantities times prices provides the labour input cost (or the labour bill, as it is sometimes called).⁶

The classes *K* and *L* concern so-called primary inputs. The letters *E*, *M*, and *S* denote three, disjunct classes of so-called intermediate inputs. First, *E* is the class of energy commodities consumed by a production unit: oil, gas, electricity, and water. Second, *M* is the class of all the (physical) materials consumed in the production process, which could be sub-classified into raw materials, semi-fabricates, and auxiliary products. Third, *S* is the class of all the business services which are consumed for maintaining the production process. Though it is not at all a trivial task to define precisely all the intermediate inputs and to classify them, it can safely be assumed that at the end of each accounting period there is a quantity and a price associated with each of those inputs.

Then, for each accounting period, production cost is defined as the sum of primary and intermediate input cost. Though this is usually not done, there are good reasons to exclude R&D

⁴ Note that the role of inventories is disregarded.

⁵ The productive capital stock may be underutilized, which implies that not all the capital costs are incurred in actual production. See Schreyer (2001, section 5.6) for a general discussion of this issue. For a treatment in the neo-classical framework the reader is referred to Berndt and Fuss (1986), Hulten (1986) and Morrison Paul (1999). We return to this issue later on.

⁶ The utilization rate of the labour input factors is assumed to be 1. Over- or underutilization from the point of view of jobs or persons is reflected in the wage rates.

expenditure from production cost, the reason being that such expenditure is not related to the current but to future output. Put otherwise, by performing R&D, production units try to shift the technology frontier. When it then comes to explaining productivity change, the non-exclusion of R&D expenditure might easily lead to a sort of double-counting error.⁷

On the output side, the letter *Y* denotes the class of commodities, goods and/or services, which are produced by the unit. Though in some industries, such as services industries or industries producing mainly unique goods, definitional problems are formidable, it can safely be assumed that for each accounting period there are data on quantities produced. For units operating on the market there are also prices. The sum of quantities times prices then provides the production revenue, and, apart from taxes on production, revenue minus cost yields profit.

Profit is an important financial performance measure. A somewhat less obvious, but equally useful, measure is ‘profitability,’ defined as revenue *divided* by cost. Profitability gives, in monetary terms, the quantity of output per unit of input, and is thus a measure of return to aggregate input (called in some older literature ‘return to the dollar’).

Monitoring the unit’s performance over time is here understood to mean monitoring the development of its profit or its profitability. Both measures are, by nature, dependent on price and quantity changes, at both sides of the unit. If there is (price) inflation and the unit’s profit has increased then that mere fact does not necessarily mean that the unit has been performing better. Also, though general inflation does not influence the development of profitability, differential inflation does. If output prices have increased more than input prices then any increase of profitability does not necessarily imply that the unit has been performing better. Thus, for measuring the economic performance of the unit, one wants to get rid of the effect of price changes.

Profit and profitability are different, but equivalent, concepts. The first is a difference measure; the second is a ratio measure. Change of a variable through time, which will be our main focus, can also be measured by a difference or a ratio. Apart from technical details -- such as, that a ratio does not make sense if the variable in the denominator changes sign or becomes equal to zero -- these two ways of measuring change are equivalent. Thus there appear to be a number of ways of mapping the same reality in numbers, and differing numbers do not necessarily imply differing realities.

Profit change stripped of its price component will be called *real* profit change, and profitability change stripped of its price component will be called *real* profitability change.⁸ Another name for real profit (-ability) change is (total factor) productivity change. Thus, productivity change can be measured as a ratio (namely as real profitability change) or as a difference (namely as real profit change). At the economy level, productivity change can be related to some measure of overall welfare change. A down-to-earth approach would use the National Accounts to establish a link between labour productivity change and real-income-per-capita change. A more sophisticated approach, using economic models and assumptions, was provided by Basu and Fernald (2002).

⁷ See Diewert and Huang (2008) for more on this issue. A big problem seems to be the separation of the R&D part of labour input.

⁸ Note that real change means nominal change deflated by some price index, not necessarily being a (headline) CPI.

For a non-market unit the story must be told somewhat differently. For such a unit, there are no output prices; hence, there is no revenue. Though there is cost, like for market units, there is no profit or profitability. National accountants usually resolve the problem here by *defining* the revenue of a non-market unit to be equal to its cost, thereby setting profit equal to 0 or profitability equal to 1.⁹ But this leaves the problem that there is no natural way of splitting revenue change through time into real and monetary components. This can only be done satisfactorily when there is some output quantity index that is independent from the input quantity index.¹⁰

It is useful to remind the reader that the notions of profit and profitability, though conceptually rather clear, are difficult to operationalize. One of the reasons is that cost includes the cost of owned capital assets, the measurement of which exhibits a substantial number of degrees of freedom, as we will see in the remainder of this paper. Also, labour cost includes the cost of self-employed persons, for which wage rates and hours of work usually must be imputed. It will be clear that all these, and many other, uncertainties spill over to operational definitions of the profit and profitability concepts.

2.1 Notation

Let us now introduce some notation to define the various concepts we are going to use. As stated, on the output side we have M items, each with their price (received) p_m^t and quantity y_m^t , where $m = 1, \dots, M$, and t denotes an accounting period (say, a year). Similarly, on the input side we have N items, each with their price (paid) w_n^t and quantity x_n^t , where $n = 1, \dots, N$. To avoid notational clutter, simple vector notation will be used throughout. All the prices and quantities are assumed to be positive, unless stated otherwise. The ex post accounting point-of-view will be used; that is, quantities and monetary values of the so-called flow variables (output and labour, energy, materials, services inputs) are realized values, complete knowledge of which becomes available not before the accounting period has expired. Similarly, the cost of capital input is calculated ex post. This is consistent with statistical practice.

The unit's revenue, that is, the value of its (gross) output, during the accounting period t is defined as

$$(1) \quad R^t \equiv p^t \cdot y^t \equiv \sum_{m=1}^M p_m^t y_m^t,$$

whereas its production cost is defined as

$$(2) \quad C^t \equiv w^t \cdot x^t \equiv \sum_{n=1}^N w_n^t x_n^t.$$

The unit's *profit* (disregarding tax on production) is then given by its revenue minus its cost; that is,

⁹ This approach goes back to Hicks (1940).

¹⁰ See the insightful paper by Douglas (2006). Though written from a New Zealand perspective, its theme is generic.

$$(3) \quad \Pi^t \equiv R^t - C^t = p^t \cdot y^t - w^t \cdot x^t.$$

The unit's profitability (also disregarding taxes on production) is defined as its revenue divided by its cost; that is,

$$(4) \quad R^t / C^t = p^t \cdot y^t / w^t \cdot x^t.$$

Notice that profitability expressed as a percentage, $R^t / C^t - 1$, equals the ratio of profit to cost, Π^t / C^t . In some circles this is called the margin of the unit. Given positive prices and quantities, it will always be the case that $R^t > 0$ and $C^t > 0$. Thus, profitability R^t / C^t is always positive, but profit $\Pi^t = R^t - C^t$ can be positive, negative, or zero.

As stated, we are concerned with intertemporal comparisons. Moreover, in this paper, only bilateral comparisons will be considered, say comparing a certain period t to another, adjacent or non-adjacent period, t' . Without loss of generality it may be assumed that period t' precedes period t . To further simplify notation, the two periods will be labeled by $t = 1$ (which will be called the comparison period) and $t' = 0$ (which will be called the base period).

2.2 Productivity Index

The development over time of profitability is, rather naturally, measured by the ratio $(R^1 / C^1) / (R^0 / C^0)$. How to decompose this into a price and a quantity component? By noticing that

$$(5) \quad \frac{R^1 / C^1}{R^0 / C^0} = \frac{R^1 / R^0}{C^1 / C^0},$$

we see that the question reduces to the question of how to decompose the revenue ratio (R^1 / R^0) and the cost ratio (C^1 / C^0) into two parts. The natural answer is to grab from the economic statistician's toolkit a pair of price and quantity indices that satisfy the Product Test:

$$(6) \quad \frac{p^1 \cdot y^1}{p^0 \cdot y^0} = P(p^1, y^1, p^0, y^0) Q(p^1, y^1, p^0, y^0).$$

The Fisher price and quantity indices are a good choice, since these indices satisfy not only the basic axioms of price and quantity measurement (see appendix A), but also a number of other relatively important requirements (such as the Time Reversal Test). Thus we are using here the 'instrumental' or 'axiomatic' approach for selecting measures for aggregate price and quantity change, an approach that goes back to Fisher (1922); see Balk (1995) for a survey and Balk (2008) for an up-to-date treatment. When the time distance between the periods 1 and 0 is not too

large, then any index that is a second-order differential approximation to the Fisher index may instead be used.¹¹

Throughout this paper, when it comes to solving problems such as (6) we will assume that Fisher indices are used. Thus, in particular,

$$(7) \quad \frac{R^1}{R^0} = P^F(p^1, y^1, p^0, y^0) Q^F(p^1, y^1, p^0, y^0) \\ \equiv P_R(1,0) Q_R(1,0),$$

where the second line serves to define our shorthand notation. In the same way, we decompose

$$(8) \quad \frac{C^1}{C^0} = P^F(w^1, x^1, w^0, x^0) Q^F(w^1, x^1, w^0, x^0) \\ \equiv P_C(1,0) Q_C(1,0).$$

Of course, the dimensionality of the Fisher indices in expressions (7) and (8) is different.

The number of items distinguished on the output side (M) and the input side (N) of a production unit can be very high. To accommodate this, (detailed) classifications are used, by which all the items are allocated to hierarchically organized (sub-)aggregates. The calculation of output and input indices then proceeds in stages. Theoretically, it suffices to distinguish only two stages. At the first stage one calculates indices for the subaggregates at some level, and at the second stage these subaggregate indices are combined to aggregate indices.

Consequentially, in expressions (7) and (8), instead of one-stage, also two-stage Fisher indices may be used; that is, Fisher indices of Fisher indices for subaggregates (see appendix A for precise definitions). Since the Fisher index is not consistent-in-aggregation, a decomposition by two-stage Fisher indices will in general numerically differ from a decomposition by one-stage Fisher indices. Fortunately, one-stage and two-stage Fisher indices are second-order differential approximations of each other (as shown by Diewert, 1978).

Using the two relations (7) and (8), the profitability ratio can be decomposed as

$$(9) \quad \frac{R^1 / C^1}{R^0 / C^0} = \frac{R^1 / R^0}{C^1 / C^0} \\ = \frac{P_R(1,0) Q_R(1,0)}{P_C(1,0) Q_C(1,0)}.$$

The (total factor) productivity index ($IPROD$), for period 1 relative to 0, is now defined by

$$(10) \quad IPROD(1,0) \equiv \frac{Q_R(1,0)}{Q_C(1,0)}.$$

¹¹ Note, however, that this is not unproblematic. For instance, when the Törnqvist price index $P^T(\cdot)$ is used, then the implicit quantity index $(p^1 \cdot y^1 / p^0 \cdot y^0) / P^T(\cdot)$ does not necessarily satisfy the Identity Test $A3'$ (see appendix A).

Thus $IPROD(1, 0)$ is the real or quantity component of the profitability ratio. Put otherwise, it is the ratio of an output quantity index to an input quantity index; $IPROD(1, 0)$ is the factor with which the output quantities on average have changed relative to the factor with which the input quantities on average have changed. If the ratio of these factors is larger (smaller) than 1, there is said to be productivity increase (decrease).¹²

Notice that, using (7) and (8), there appear to be three other equivalent representations of the productivity index, namely

$$(11) \quad IPROD(1,0) = \frac{(R^1 / R^0) / P_R(1,0)}{(C^1 / C^0) / P_C(1,0)}$$

$$(12) \quad = \frac{(R^1 / R^0) / P_R(1,0)}{Q_C(1,0)}$$

$$(13) \quad = \frac{Q_R(1,0)}{(C^1 / C^0) / P_C(1,0)} .$$

Put in words, we are seeing here respectively a deflated revenue index divided by a deflated cost index, a deflated revenue index divided by an input quantity index, and an output quantity index divided by a deflated cost index.

Further, if the revenue change equals the cost change, $R^1 / R^0 = C^1 / C^0$ (for which zero profit in the two periods is a sufficient condition), then it follows from (9) that

$$(14) \quad IPROD(1,0) = \frac{P_C(1,0)}{P_R(1,0)} ;$$

that is, the productivity index is equal to an input price index divided by an output price index. In general, however, the dual productivity index $P_C(1,0) / P_R(1,0)$ will differ from the primal one, $Q_R(1,0) / Q_C(1,0)$.

2.3 Growth Accounting

The foregoing definitions are already sufficient to provide examples of simple but useful analysis. Consider relation (12), and rewrite this as

$$(15) \quad R^1 / R^0 = IPROD(1,0) \times Q_C(1,0) \times P_R(1,0) .$$

Recall that revenue change through time is only interesting in so far as it differs from general inflation. Hence, it makes sense to deflate the revenue index, R^1 / R^0 , by a general inflation measure such as the Consumer Price Index (CPI). Doing this, the last equation becomes

$$(16) \quad \frac{R^1 / R^0}{CPI^1 / CPI^0} = IPROD(1,0) \times Q_C(1,0) \times \frac{P_R(1,0)}{CPI^1 / CPI^0} .$$

¹² This approach follows Diewert (1992), Diewert and Nakamura (2003), and Balk (2003).

Lawrence, Diewert and Fox (2006) basically use this relation to decompose ‘real’ revenue change into three factors: productivity change, input quantity change (which can be interpreted as measuring change of the unit’s size), and ‘real’ output price change respectively.

Another example follows from rearranging expression (13) and taking logarithms. This delivers the following relation:

$$(17) \quad \ln\left(\frac{C^1/C^2}{Q_R(1,0)}\right) = \ln P_C(1,0) - \ln IPROD(1,0).$$

We see here that the growth rate of average cost can be decomposed into two factors, namely the growth rate of input prices and a residual which is the negative of productivity growth. Put otherwise, in the case of stable input prices the growth rate of average cost is equal to minus the productivity growth rate.

Both are examples of what is called *growth accounting*. The relationship between index number techniques and growth accounting can, more generally, be seen as follows. Recall the generic definition (10), and rewrite this expression as

$$(18) \quad Q_R(1,0) = IPROD(1,0) \times Q_C(1,0).$$

Using logarithms, this multiplicative expression can be rewritten as

$$(19) \quad \ln Q_R(1,0) = \ln IPROD(1,0) + \ln Q_C(1,0).$$

For index numbers in the neighborhood of 1, the logarithms thereof reduce to percentages, and the last expression can be interpreted as saying that the percentage change of output volume equals the percentage change of input volume plus the percentage change of productivity. Growth accounting economists like to work with equations expressing output volume growth in terms of input volume growth plus a residual that is interpreted as productivity growth, thereby suggesting that the last two factors cause the first. However, productivity change cannot be considered as an independent factor since it is *defined* as output quantity change minus input quantity change. A growth accounting table is just an alternative way of presenting productivity growth and its contributing factors. And decomposition does not imply anything about causality.¹³

2.4 Productivity Indicator

Let us now turn to profit and its development through time. This is naturally measured by the difference $\Pi^1 - \Pi^0$. Of course, such a difference only makes sense when the two money amounts involved, profit from period 0 and profit from period 1, are deflated by some general inflation measure (such as the headline CPI). In the remainder of this paper, when discussing difference measures, such a deflation is tacitly presupposed.

How to decompose the profit difference into a price and a quantity component? By noticing that

¹³ Thus, saying that output growth outpaced input growth because TFP increased is “like saying that the sun rose because it was morning”, to paraphrase Friedman (1988, p. 58).

$$(20) \quad \Pi^1 - \Pi^0 = (R^1 - R^0) - (C^1 - C^0),$$

we see that the question reduces to the question of how to decompose revenue change ($R^1 - R^0$) and cost change ($C^1 - C^0$) into two parts. We now grab from the economic statistician's toolkit a pair of price and quantity indicators that satisfy the Product Test:

$$(21) \quad p^1 \cdot y^1 - p^0 \cdot y^0 = P(p^1, y^1, p^0, y^0) + Q(p^1, y^1, p^0, y^0).$$

A good choice is the Bennet (1920) price and quantity indicator, since these indicators satisfy not only the basic axioms (see appendix A), but also a number of other relatively important requirements (such as the Time Reversal Test) (see Diewert 2005). But any indicator that is a second-order differential approximation to the Bennet indicator may instead be used. Thus,

$$(22) \quad \begin{aligned} R^1 - R^0 &= P^B(p^1, y^1, p^0, y^0) + Q^B(p^1, y^1, p^0, y^0) \\ &\equiv P_R(1,0) + Q_R(1,0), \end{aligned}$$

and similarly,

$$(23) \quad \begin{aligned} C^1 - C^0 &= P^B(w^1, x^1, w^0, x^0) + Q^B(w^1, x^1, w^0, x^0) \\ &\equiv P_C(1,0) + Q_C(1,0). \end{aligned}$$

Notice that the dimensionality of the Bennet indicators in these two decompositions is in general different.

The Bennet indicators are difference analogs to Fisher indices. Their aggregation properties, however, are much simpler. The Bennet price or quantity indicator for an aggregate is equal to the sum of the subaggregate indicators.

Using indicators, the profit difference can be written as

$$(24) \quad \begin{aligned} \Pi^1 - \Pi^0 &= P_R(1,0) + Q_R(1,0) - [P_C(1,0) + Q_C(1,0)] \\ &= P_R(1,0) - P_C(1,0) + Q_R(1,0) - Q_C(1,0). \end{aligned}$$

The first two terms at the right-hand side of the last equality sign provide the price component, whereas the last two terms provide the quantity component of the profit difference. Thus, based on this decomposition, the (total factor) productivity indicator (*DPROD*) is defined by

$$(25) \quad DPROD(1,0) \equiv Q_R(1,0) - Q_C(1,0);$$

that is, an output quantity indicator minus an input quantity indicator. Notice that productivity change is now measured as an amount of money. An amount larger (smaller) than 0 indicates productivity increase (decrease).¹⁴

The equivalent expressions for difference-type productivity change are

$$(26) \quad DPROD(1,0) = [R^1 - R^0 - P_R(1,0)] - [C^1 - C^0 - P_C(1,0)]$$

¹⁴ This approach follows Balk (2003).

$$(27) \quad = [R^1 - R^0 - P_R(1,0)] - Q_C(1,0)$$

$$(28) \quad = Q_R(1,0) - [C^1 - C^0 - P_C(1,0)],$$

which can be of use in different situations. Notice further that, if $R^1 - R^0 = C^1 - C^0$, then

$$(29) \quad DPROD(1,0) \equiv P_C(1,0) - P_R(1,0).$$

For a non-market production unit, a productivity indicator is difficult to define. Though one might be able to construe an output quantity indicator, it is hard to see how, in the absence of output prices, such an indicator could be given a money dimension.

2.5 Partial Productivity Measures

The productivity index $IPROD(1,0)$ and the indicator $DPROD(1,0)$ bear the adjective ‘total factor’ because all the inputs are taken into account. To define partial productivity measures, in ratio or difference form, additional notation is necessary.

All the items on the input side of our production unit are assumed to be allocatable to the five, mutually disjunct, categories mentioned earlier, namely capital (K), labour (L), energy (E), materials (M), and services (S). The entire input price and quantity vectors can then be partitioned as $w^t = (w_K^t, w_L^t, w_E^t, w_M^t, w_S^t)$ and $x^t = (x_K^t, x_L^t, x_E^t, x_M^t, x_S^t)$ respectively. Energy, materials and services together form the category of intermediate inputs, that is, inputs (denoted by EMS), which are acquired from other production units or imported. Capital and labour are called primary inputs. Consistent with this distinction the price and quantity vectors can also be partitioned as $w^t = (w_{KL}^t, w_{EMS}^t)$ and $x^t = (x_{KL}^t, x_{EMS}^t)$, or as $w^t = (w_K^t, w_L^t, w_{EMS}^t)$ and $x^t = (x_K^t, x_L^t, x_{EMS}^t)$. Since monetary values are additive, total production cost can be decomposed in a number of ways, such as

$$(30) \quad \begin{aligned} C^t &= \sum_{n \in K} w_n^t x_n^t + \sum_{n \in L} w_n^t x_n^t + \sum_{n \in E} w_n^t x_n^t + \sum_{n \in M} w_n^t x_n^t + \sum_{n \in S} w_n^t x_n^t \\ &\equiv C_K^t + C_L^t + C_E^t + C_M^t + C_S^t \\ &\equiv C_K^t + C_L^t + C_{EMS}^t \\ &\equiv C_{KL}^t + C_{EMS}^t. \end{aligned}$$

Now, using as before Fisher indices, the labour cost ratio can be decomposed as

$$(31) \quad \begin{aligned} \frac{C_L^1}{C_L^0} &= P^F(w_L^1, x_L^1, w_L^0, x_L^0) Q^F(w_L^1, x_L^1, w_L^0, x_L^0) \\ &\equiv P_L(1,0) Q_L(1,0). \end{aligned}$$

Then the labour productivity index ($ILPROD$) for period 1 relative to period 0 is defined by

$$(32) \quad ILPROD(1,0) \equiv \frac{Q_R(1,0)}{Q_L(1,0)};$$

that is, the ratio of an output quantity index to a labour input quantity index. Notice that usually the labour productivity index is defined by specifying the labour input quantity index to be the Dutot or simple sum quantity index $Q^D(w_L^1, x_L^1, w_L^0, x_L^0) \equiv \sum_{n \in L} x_n^1 / \sum_{n \in L} x_n^0$. The ratio $Q^F(w_L^1, x_L^1, w_L^0, x_L^0) / Q^D(w_L^1, x_L^1, w_L^0, x_L^0)$ is then said to measure the shift in labour quality or composition.

In precisely the same way, one can define the capital productivity index

$$(33) \quad IKPROD(1,0) \equiv \frac{Q_R(1,0)}{Q_K(1,0)},$$

and the other partial productivity indices $IkPROD$ for $k = E, M, S$. The ratio

$$(34) \quad \frac{ILPROD(1,0)}{IKPROD(1,0)} = \frac{Q_K(1,0)}{Q_L(1,0)}$$

is called the index of ‘capital deepening’. Loosely speaking, this index measures the change of the quantity of capital input per unit of labour input.

The relation between total factor and partial productivity indices is as follows. Let $Q_C(1,0)$ be a two-stage Fisher index; that is,

$$(35) \quad Q_C(1,0) \equiv Q^F(Q_k(1,0), C_k^1, C_k^0; k = K, L, E, M, S),$$

where all the $Q_k(1,0)$ are Fisher indices. It is straightforward to check that then

$$(36) \quad \begin{aligned} IPROD(1,0) &= \frac{Q_R(1,0)}{Q_C(1,0)} \\ &= \frac{Q_R(1,0)}{\left(\sum_k Q_k(1,0) C_k^0 / C^0 \right)^{1/2} \left(\sum_k Q_k(1,0)^{-1} C_k^1 / C^1 \right)^{-1/2}} \\ &= \left(\sum_k \frac{Q_k(1,0)}{Q_R(1,0)} \frac{C_k^0}{C^0} \right)^{-1/2} \left(\sum_k \frac{Q_R(1,0)}{Q_k(1,0)} \frac{C_k^1}{C^1} \right)^{1/2} \\ &= \left(\frac{\sum_k C_k^0 (IkPROD(1,0))^{-1}}{C^0} \right)^{-1/2} \left(\frac{\sum_k C_k^1 IkPROD(1,0)}{C^1} \right)^{-1/2}, \end{aligned}$$

which is not a particularly simple relation. If instead as a second-stage quantity index the Cobb-Douglas functional form was chosen, that is,

$$(37) \quad Q_C(1,0) \equiv \prod_k Q_k(1,0)^{\alpha_k} \quad \text{where } \sum_k \alpha_k = 1 \quad (\alpha_k > 0),$$

then it appears that

$$(38) \quad \ln IPROD(1,0) = \sum_k \alpha_k \ln IkPROD(1,0).$$

This is a very simple relation between total factor productivity change and partial productivity change. Notice, however, that this simplicity comes at a cost. Definition (37) implies for the relation between aggregate and subaggregate input price indices that

$$(39) \quad P_C(1,0) = \prod_k P_k(1,0)^{\alpha_k} \frac{C^1/C^0}{\prod_k (C_k^1/C_k^0)^{\alpha_k}}.$$

Such an index does not necessarily satisfy the fundamental Identity Test A3; that is, if all prices are the same in period 1 as in period 0, then $P_C(1,0)$ does not necessarily deliver the outcome 1.

Let us now turn to partial productivity *indicators*. Using the Bennet indicators, the labour cost difference between periods 0 and 1 is decomposed as

$$(40) \quad \begin{aligned} C_L^1 - C_L^0 &= P^B(w_L^1, x_L^1, w_L^0, x_L^0) + Q^B(w_L^1, x_L^1, w_L^0, x_L^0) \\ &\equiv P_L(1,0) + Q_L(1,0). \end{aligned}$$

Similarly, one can decompose the capital, energy, materials, and services cost difference. However, costs are additive, the total factor productivity indicator can be written as

$$(41) \quad DPROD(1,0) = Q_R(1,0) - \sum_{k=K,L,E,M,S} Q_k(1,0).$$

By definition, the left-hand side is real profit change. The right-hand side gives the contributing factors. The contribution of category k to real profit change is simply measured by the amount $Q_k(1,0)$. A positive amount, which means that the aggregate quantity of input category k has increased, means a negative contribution to real profit change.

3. Different Models, Similar Measures

The previous section laid out the basic features of what is known as the KLEMS model of production. This framework is currently used by the U.S. Bureau of Labor Statistics (BLS) and Statistics Canada for productivity measures at the industry level of aggregation (see Dean and Harper 2001, and Harchaoui *et al.* 2001 respectively). The KLEMS model, or, as I will denote it, the KLEMS-Y model delivers gross-output based total or partial productivity measures. However, there are more models in use, differing from the KLEMS-Y model by their input and output concepts. Since these models presuppose revenue as measured independently from cost, they are not applicable to non-market units.

3.1 The KL-VA model

The first of these models uses value added (VA) as its output concept. The production unit's value added (VA) is defined as its revenue minus the costs of energy, materials, and services; that is,

$$(42) \quad \begin{aligned} VA^t &\equiv R^t - C_{EMS}^t \\ &= p^t \cdot y^t - w_{EMS}^t \cdot x_{EMS}^t. \end{aligned}$$

The value-added concept subtracts the total cost of intermediate inputs from the revenue obtained, and in doing so essentially conceives the unit as producing value added (that is, money) from the two primary input categories capital and labour. It is assumed that $VA^t > 0$.

Although gross output, represented by y^t , is the natural output concept, the value-added concept is important when one wishes to aggregate single units to larger entities. Gross output consists of deliveries to final demand and intermediate destinations. The split between these two output categories depends very much on the level of aggregation. Value added is immune to this problem. It enables one to compare (units belonging to) different industries. From a welfare-theoretic point of view the value-added concept is important because value added can be conceived of as the income (from production) that flows into society.¹⁵

In this input-output model the counterpart to *profitability* is the ratio of value added to primary inputs cost, VA^t / C_{KL}^t , and the natural starting point for defining a productivity index is to consider the development of this ratio through time. Since $(VA^1 / C_{KL}^1) / (VA^0 / C_{KL}^0) = (VA^1 / VA^0) / (C_{KL}^1 / C_{KL}^0)$, we need a decomposition of the value-added ratio and a decomposition of the primary inputs cost ratio.

The question of how to decompose a value-added ratio into price and quantity components cannot be answered unequivocally. There are several options here, the technical details of which are deferred to appendix B. Suppose, however, that a satisfactory decomposition is somehow available; that is,

$$(43) \quad \frac{VA^1}{VA^0} = P_{VA}(1,0)Q_{VA}(1,0).$$

Using one- or two-stage Fisher indices, the primary inputs cost ratio is decomposed as

$$(44) \quad \begin{aligned} \frac{C_{KL}^1}{C_{KL}^0} &= P^F(w_{KL}^1, x_{KL}^1, w_{KL}^0, x_{KL}^0)Q^F(w_{KL}^1, x_{KL}^1, w_{KL}^0, x_{KL}^0) \\ &\equiv P_{KL}(1,0)Q_{KL}(1,0). \end{aligned}$$

The value-added based (total factor) productivity index for period 1 relative to period 0 is then defined as

$$(45) \quad IPROD_{VA}(1,0) \equiv \frac{Q_{VA}(1,0)}{Q_{KL}(1,0)}.$$

¹⁵ In between the KLEMS-Y model and the KL-VA model there is the KLE”M”S-Margin model, applicable to distributive trade units. Here the set of material inputs M is split into two parts, M’ denoting the goods for resale and M” the auxiliary materials. Likewise E, the set of energy inputs, is split into E’ and E”. The Margin is then defined as revenue minus the cost of the goods for resale and the cost of the first of the two energy components. See Inklaar and Timmer (2008).

This index measures the ‘quantity’ change of value added relative to the quantity change of primary input; or, can be seen as the index of real value added relative to the index of real primary input.

This is by far the most common model. It is used by the U. S. Bureau of Labor Statistics, Statistics Canada, the Australian Bureau of Statistics, Statistics New Zealand, and the Swiss Federal Statistical Office in their official productivity statistics.

In the KL-VA model the counterpart to *profit* is the difference of value added and primary inputs cost, $VA^t - C_{KL}^t$, and the natural starting point for defining a productivity indicator is to consider the development of this difference through time. However, since costs are additive, we see that, by using definition (42),

$$(46) \quad \begin{aligned} VA^t - C_{KL}^t &= R^t - C_{EMS}^t - C_{KL}^t \\ &= R^t - C^t. \end{aligned}$$

Thus, profit in the KL-VA model is the same as profit in the KLEMS-Y model, and the same applies to the price and quantity components of profit differences. Using Bennet indicators, one can easily check that

$$(47) \quad \begin{aligned} DPROD_{VA}(1,0) &\equiv Q_{VA}(1,0) - Q_{KL}(1,0) \\ &= Q_R(1,0) - Q_C(1,0) \\ &= DPROD(1,0); \end{aligned}$$

that is, the productivity indicators are the same in the two models. This, however, does not hold for the productivity indices. One usually finds that $IPROD_{VA}(1,0) \neq IPROD(1,0)$. Balk (2009)

showed that if profit is zero in both periods, that is $R^t = C^t$ ($t = 0,1$), then, for certain two-stage indices which are second-order differential approximations to Fisher indices,

$$(48) \quad \ln IPROD_{VA}(1,0) = D(1,0) \ln IPROD(1,0),$$

where $D(1,0) \geq 1$ is the (mean) Domar-factor (= ratio of revenue over value added). Usually expression (48) is, in a continuous-time setting, derived under a set of strong neo-classical assumptions (see, for instance, Gollop (1979), Jorgenson *et al.* (2005, p. 298) or Schreyer (2001, p. 143)), so that it seems to be some deep economic-theoretical result. From the foregoing it may be concluded, however, that the inequality of the value-added based productivity index and the gross-output based productivity index is only due to the mathematics of ratios and differences. It does not point to any underlying economic phenomenon.

The value-added based labour productivity index for period 1 versus 0 is defined as

$$(49) \quad ILPROD_{VA}(1,0) \equiv \frac{Q_{VA}(1,0)}{Q_L(1,0)},$$

where $Q_L(1,0)$ was defined by expression (31). The index defined by expression (49) measures the ‘quantity’ change of value added relative to the quantity change of labour input; or, can be seen as the index of real value added relative to the index of real labour input.

Recall that the labour quantity index $Q_L(1,0)$ is here defined as a Fisher index, acting on the prices and quantities of all the types of labour that are being distinguished. Suppose that the units of measurement of the various types are in some sense the same; that is, the quantities of all the types of labour are measured in hours, or in full-time equivalent jobs, or in some other common unit. Then one frequently considers, instead of the Fisher quantity index, the Dutot or simple sum quantity index,

$$(50) \quad Q_L^D(1,0) \equiv \sum_{n \in L} x_n^1 / \sum_{n \in L} x_n^0 .$$

The simple value-added based labour productivity index, defined as

$$(51) \quad ILPROD_{VA}^D(1,0) \equiv \frac{Q_{VA}(1,0)}{Q_L^D(1,0)} ,$$

has the alternative interpretation as an index of real value added per unit of labour. As such this measure frequently figures at the left-hand side (thus, as *explanandum*) in a growth accounting equation. However, for deriving such a relation nothing spectacular is needed, as will now be shown.

Consider the definition of the value-added based total factor productivity index, (45), and rewrite this as

$$(52) \quad Q_{VA}(1,0) = IPROD_{VA}(1,0) \times Q_{KL}(1,0) .$$

Dividing both sides of this equation by the Dutot labour quantity index, and applying definition (51), one obtains¹⁶

$$(53) \quad ILPROD_{VA}^D(1,0) = IPROD_{VA}(1,0) \times \frac{Q_{KL}(1,0)}{Q_L(1,0)} \times \frac{Q_L(1,0)}{Q_L^D(1,0)} .$$

Taking logarithms and, on the assumption that all the index numbers are in the neighborhood of 1, interpreting these as percentages, the last equation can be interpreted as: (simple) labour productivity growth equals total factor productivity growth plus ‘capital deepening’ plus ‘labour quality’ growth. Again, productivity change is measured as a residual and, thus, the three factors at the right-hand side of the last equation can in no way be regarded as causal factors.

If, continuing our previous example, the primary inputs quantity index was defined as a two-stage index of the form

$$(54) \quad Q_{KL}(1,0) \equiv Q_K(1,0)^\alpha Q_L(1,0)^{1-\alpha} , \quad \text{for } 0 < \alpha < 1 ,$$

where the reader recognizes the simple Cobb-Douglas form, then the index of ‘capital deepening’ reduces to the particularly simple form

$$(55) \quad \frac{Q_{KL}(1,0)}{Q_L(1,0)} = \left[\frac{Q_K(1,0)}{Q_L(1,0)} \right]^\alpha .$$

¹⁶ This is a discrete time version of expression (23) of Baldwin, Gu and Yan (2007).

The ‘labour quality’ index, $Q_L(1,0)/Q_L^D(1,0)$, basically measures compositional shift or structural change among the labour types in the class L , because it is a ratio of two quantity indices.

3.2 The K-CF model

The next model uses cash flow (CF) as its output concept.¹⁷ The unit’s cash flow is defined as its revenue minus the costs of labour and intermediate inputs; that is

$$\begin{aligned}
 (56) \quad CF^t &\equiv R^t - C_{LEMS}^t \\
 &= p^t \cdot y^t - w_{LEMS}^t \cdot x_{LEMS}^t \\
 &= VA^t - C_L^t.
 \end{aligned}$$

This input-output model basically sees cash flow as the return to capital input. It is assumed that $CF^t > 0$. Of course, if there is no owned capital (that is, all capital assets are leased), then $C_K^t = 0$, and this model does not make sense.

The counterpart to *profitability* is now the ratio of cash flow to capital input cost, CF^t / C_K^t , and the natural starting point for defining a productivity index is to consider the development of this ratio through time. Since $(CF^1 / C_K^1) / (CF^0 / C_K^0) = (CF^1 / CF^0) / (C_K^1 / C_K^0)$, we need a decomposition of the cash-flow ratio and a decomposition of the capital input cost ratio.

Decomposing a cash-flow ratio into price and quantity components is structurally similar to decomposing a value-added ratio (see appendix B). Thus, suppose that a satisfactory decomposition is somehow available; that is,

$$(57) \quad \frac{CF^1}{CF^0} = P_{CF}(1,0)Q_{CF}(1,0).$$

Using Fisher indices, the capital input cost ratio is decomposed as

$$\begin{aligned}
 (58) \quad \frac{C_K^1}{C_K^0} &= P^F(w_K^1, x_K^1, w_K^0, x_K^0)Q^F(w_K^1, x_K^1, w_K^0, x_K^0) \\
 &\equiv P_K(1,0)Q_K(1,0).
 \end{aligned}$$

The cash-flow based (total factor) productivity index for period 1 versus 0 is then defined as

$$(59) \quad IPROD_{CF}(1,0) = \frac{Q_{CF}(1,0)}{Q_K(1,0)}.$$

¹⁷ Cash flow is also called gross profit. The National Accounts term is ‘gross operating surplus.’

This index measures the change of the quantity component of cash flow relative to the quantity change of capital input; or, it can be seen as the index of real cash flow relative to the index of real capital input.

In the K-CF model, the counterpart to *profit* is the difference of cash flow and capital input cost, $CF^t - C_K^t$, and the natural starting point for defining a productivity indicator is to consider the development of this difference through time. However, since costs are additive, we see that

$$(60) \quad \begin{aligned} CF^t - C_K^t &= R^t - C_{LEMS}^t - C_K^t \\ &= R^t - C^t. \end{aligned}$$

Thus, profit in the K-CF model is the same as profit in the KLEMS-Y model, and the same applies to the price and quantity components of profit differences. Using Bennet indicators, one easily checks that

$$(61) \quad \begin{aligned} DPROD_{CF}(1,0) &\equiv Q_{CF}(1,0) - Q_K(1,0) \\ &= Q_R(1,0) - Q_C(1,0) \\ &= DPROD(1,0); \end{aligned}$$

that is, the productivity indicators are the same in the two models. This, however, does not hold for the productivity indices. In general it will be the case that $DPROD_{CF}(1,0) \neq DPROD(1,0)$. Following the reasoning of Balk (2009), it is possible to show that, if profit is zero in both periods, that is, $R^t = C^t$ ($t = 0,1$), then, for certain two-stage indices which are second-order differential approximations to Fisher indices,

$$(62) \quad \ln IPROD_{CF}(1,0) = E(1,0) \ln IPROD(1,0),$$

where $E(1,0) \geq 1$ is the ratio of mean revenue over mean cash flow. Since $CF^t \leq VA^t$, it follows that $E(1,0) \geq D(1,0)$.

4. Capital Input Cost

The K-CF model provides a good point of departure for a discussion of the measurement of capital input cost. Cash flow, as defined in the foregoing, is the (ex post measured) monetary balance of all the flow variables. Capital input cost is different, since capital is a stock variable. Basically, capital input cost is measured as the difference between the book values of the production unit's owned capital stock at beginning and end of the accounting period considered.

Our notation must therefore be extended. The beginning of period t is denoted by t^- , and its end by t^+ . Thus a period is an interval of time $t = [t^-, t^+]$, where $t^- = (t-1)^+$ and $t^+ = (t+1)^-$. Occasionally, the variable t will also be used to denote the midpoint of the period.

All the assets are supposed to be economically born at midpoints of periods, whether this

has occurred inside or outside the production unit under consideration. Thus the age of an asset of type i at (the midpoint of) period t is a non-negative integer number $j = 0, \dots, J_i$. The age of this asset at the beginning of the period is $j - 0.5$, and at the end is $j + 0.5$. The economically maximal service life of asset type i is denoted by J_i .

The opening stock of capital assets is the inheritance of past investments and disinvestments; hence, this stock consists of cohorts of assets of various types, each cohort comprising a number of assets of the same age. By (Netherlands' National Accounts) convention, assets that are discarded (normally retired or prematurely scrapped) or sold during a certain period t are supposed to be discarded or sold at the end of that period; that is, at t^+ . Second-hand assets that are acquired during period t from other production units are supposed to be acquired at the beginning of the next period, $(t+1)^-$. However, all other acquisitions of second-hand assets and those of new assets are supposed to happen at the midpoint of the period, and to be immediately operational.

Hence, all the assets that are part of the opening stock remain active through the entire period $[t^-, t^+]$. The period t investments are supposed to be active through the second half of period t , that is, $[t, t^+]$. Put otherwise, the stock of capital assets at t , the midpoint of the period, is the same as the stock at t^- , the beginning of the period, but 0.5 period older. At the midpoint of the period the investments, of various age, are added to the stock. Notice, however, that the closing stock at t^+ , the end of the period, is not necessarily identical to the opening stock at $(t+1)^-$, because of the conventions regarding the sale, acquisition, and discard of assets.

Let K_{ij}^t denote the quantity (number) of asset type i ($i = 1, \dots, I$) and age j ($j = 1, \dots, J_i$) at the midpoint of period t . These quantities are nonnegative; some of them might be equal to 0. Further, let I_{ij}^t denote the (non-negative) quantity (number) of asset type i ($i = 1, \dots, I$) and age j ($j = 0, \dots, J_i$) that is added to the stock at the midpoint of period t . The following relations are useful to keep in mind:

$$(63) \quad K_{i,j-0.5}^{t^-} = K_{ij}^t \quad \text{for } j = 1, \dots, J_i,$$

$$(64) \quad I_{i0}^t = K_{i,0.5}^{t^+},$$

$$(65) \quad I_{ij}^t + K_{ij}^t = K_{i,j+0.5}^{t^+} \quad \text{for } j = 1, \dots, J_i,$$

$$(66) \quad K_{i,(j+1)-0.5}^{(t+1)^-} = K_{i,j+0.5}^{t^+} + B_{i,j+0.5}^{t^+} \quad \text{for } j = 1, \dots, J_i - 1,$$

and

$$(67) \quad K_{i,(J_i+1)-0.5}^{(t+1)^-} = 0,$$

where $B_{i,j+0.5}^{t+}$ denotes the balance of sale, acquisition, and discard at t^+ . We are now ready to define the concept of user cost for assets that are owned by the production unit.¹⁸

The first distinction that must be made is between assets that are part of the opening stock of a period, and investments that are made during this period. Consider an asset of type i that has age j at the midpoint of period t . Its price (or valuation) at the beginning of the period is denoted by $P_{i,j-0.5}^{t-}$, and its price (or valuation) at the end of the period by $P_{i,j+0.5}^{t+}$. For the time being, we consider such prices as being given, and postpone their precise definition to the next section. The prices are assumed to be non-negative; some might be equal to 0. In any case, $P_{i,J_i+0.5}^{t+} = 0$; that is, an asset that has reached its economically maximal age in period t is valued with a zero price at the end of this period.

The (ex post) unit user cost over period t of an opening stock asset of type i that has age j at the midpoint of the period is then *defined* as

$$(68) \quad u_{ij}^t \equiv r^t P_{i,j-0.5}^{t-} + (P_{i,j-0.5}^{t-} - P_{i,j+0.5}^{t+}) + \tau_{ij}^t, \quad \text{for } j = 1, \dots, J_i.$$

There are three components here. The first, $r^t P_{i,j-0.5}^{t-}$, is the price (or valuation) of this asset at the beginning of the period, when its age is $j-0.5$, times an interest rate. This component reflects the premium that must be paid to the owner of the asset to prevent it from being sold, right at the beginning of the period, and the revenue used for immediate consumption; it is therefore also called the price of ‘waiting.’¹⁹ Another interpretation is to see this component as the actual or imputed interest cost to finance the monetary capital that is tied up in the asset; it is then called ‘opportunity cost.’ Anyway, it is a sort of remuneration which, since there might be a risk component involved, is specific for the production unit.²⁰

The second part of expression (68), $P_{i,j-0.5}^{t-} - P_{i,j+0.5}^{t+}$, is the value change of the asset between the beginning and end of the accounting period. It is called (nominal) time-series depreciation, and combines the effect of the progress of time, from t^- to t^+ , with the effect of ageing, from $j-0.5$ to $j+0.5$. In general, the difference between the two prices (valuations) comprises the effect of exhaustion, deterioration, and obsolescence.

The third component, τ_{ij}^t , denotes the specific tax(es) that is (are) levied on the use of an asset of type i and age j during period t .

¹⁸ If there were no transactions in second-hand assets, then the number of assets K_{ij}^t would be equal to the number of new investments of j periods earlier, I_{i0}^{t-j} , adjusted for the probability of survival.

¹⁹ According to Rymes (1983) this naming goes back to Pigou.

²⁰ The System of National Accounts 1993 prescribes that for non-market units belonging to the government sector the interest rate r^t must be set equal to 0.

Unit user cost as defined in expression (68) is also called ‘rental price,’ because it can be considered as the rental price that the owner of the asset would charge to the owner as user. Put otherwise, unit user cost is like a lease price.

Let us now turn to the unit user cost of an asset of type i and age j that is acquired at the midpoint of period t . To keep things simple, this user cost is, analogous to expression (68), defined as

$$(69) \quad v_{ij}^t \equiv (1/2)r^t P_{ij}^t + (P_{ij}^t - P_{i,j+0.5}^{t+}) + (1/2)\tau_{ij}^t, \quad \text{for } j = 0, \dots, J_i.$$

The difference from the previous formula is that here the second half of the period instead of the entire period is taken into account.²¹

Total user cost over all asset types and ages, for period t , is then naturally defined by

$$(70) \quad C_K^t \equiv \sum_{i=1}^I \sum_{j=1}^{J_i} u_{ij}^t K_{ij}^t + \sum_{i=1}^I \sum_{j=0}^{J_i} v_{ij}^t I_{ij}^t.$$

The set of quantities $\{K_{ij}^t, I_{ij}^t; i = 1, \dots, I; j = 0, \dots, J_i\}$ represents the so-called productive capital stock of the production unit. This is an enumeration of the assets that make production possible. The total value of these assets at the midpoint of period t can be calculated as

$$(71) \quad \sum_{i=1}^I \sum_{j=1}^{J_i} P_{ij}^t K_{ij}^t + \sum_{i=1}^I \sum_{j=0}^{J_i} P_{ij}^t I_{ij}^t.$$

This value is called the net (or wealth) capital stock.²²

We are now able to connect the variables in expression (70) with the notation introduced in the foregoing; see expression (30). We see that the set K consists of two subsets, corresponding respectively to the type-age classes of assets that are part of the opening stock and the type-age classes of assets that are acquired later. The dimension of the first set is $\sum_{i=1}^I J_i$, and the dimension of the second set is $\sum_{i=1}^I (1 + J_i)$. The input prices w_n^t ($n \in K$) are given by expressions (68) and (69) respectively, while the quantities x_n^t ($n \in K$) are given by K_{ij}^t and I_{ij}^t respectively.

If all the variables occurring in expression (70) were observable, then our story would almost end here. However, this is not the case. Though the quantity variables are in principle observable, the price variables are not. To start with, the expressions (68) and (69) contain prices (valuations) for all asset types and ages, but, except for new assets and where markets for second-hand assets exist, these prices are not observable. Thus, we need models.

²¹ The factor $(1/2)r^t$ is meant as an approximation to $(1+r^t)^{1/2} - 1$, and the factor $(1/2)\tau_{ij}^t$ as an approximation to

$$((1 + \tau_{ij}^t / P_{i,j-0.5}^{t-})^{1/2} - 1)P_{i,j}^t.$$

²² Coremberg (2008) considers the difference between a quantity index based on expression (70) – capital services – and on expression (71) – capital stock.

5. The Relationship between Asset Price and Unit User Cost

Consider expression (68) and rewrite it in the form

$$(72) \quad u_{ij}^t - \tau_{ij}^t = (1 + r^t)P_{i,j-0.5}^{t-} - P_{i,j+0.5}^{t+} \quad \text{for } j = 1, \dots, J_i.$$

For any asset that is not prematurely discarded it will be the case that its value at the end of period t is equal to its value at the beginning of period $t+1$; formally, $P_{i,j+0.5}^{t+} = P_{i,(j+1)-0.5}^{(t+1)-}$. Substituting this into expression (72), and rewriting again, one obtains

$$(73) \quad P_{i,j-0.5}^{t-} = \frac{1}{1 + r^t} (P_{i,(j+1)-0.5}^{(t+1)-} + u_{i,j}^t - \tau_{i,j}^t) \quad \text{for } j = 1, \dots, J_i.$$

This expression links the price of an asset at the beginning of period t with its price at the beginning of period $t+1$, being then 1 period older. But a similar relation links its price at the beginning of period $t+1$ with its price at the beginning of period $t+2$, being then again 1 period older,

$$(74) \quad P_{i,(j+1)-0.5}^{(t+1)-} = \frac{1}{1 + r^{t+1}} (P_{i,(j+2)-0.5}^{(t+2)-} + u_{i,j+1}^{t+1} - \tau_{i,j+1}^{t+1}) \quad \text{for } j = 1, \dots, J_i.$$

This can be continued until

$$(75) \quad P_{i,J_i-0.5}^{(t+J_i-j)-} = \frac{1}{1 + r^{t+J_i-j}} (P_{i,J_i+0.5}^{(t+J_i-j+1)-} + u_{i,J_i}^{t+J_i-j} - \tau_{i,J_i}^{t+J_i-j}) \quad \text{for } j = 1, \dots, J_i,$$

since we know that $P_{i,J_i+0.5}^{(t+J_i-j+1)-} = P_{i,J_i+0.5}^{(t+J_i-j)+} = 0$. Substituting expression (74) into (73), etcetera, one finally obtains

$$(76) \quad P_{i,j-0.5}^{t-} = \frac{u_{i,j}^t - \tau_{i,j}^t}{1 + r^t} + \frac{u_{i,j+1}^{t+1} - \tau_{i,j+1}^{t+1}}{(1 + r^t)(1 + r^{t+1})} + \dots + \frac{u_{i,J_i}^{t+J_i-j} - \tau_{i,J_i}^{t+J_i-j}}{(1 + r^t) \dots (1 + r^{t+J_i-j})}.$$

This is a materialization of the so-called fundamental asset price equilibrium equation. Notice, however, that there is no equilibrium assumed here -- whatever that may mean -- and there are no other economic behavioral assumptions involved; it is just a mathematical result. Expressions (72) and (76) are dual. The first derives the (ex tax) unit user cost from discounted asset prices, while the second derives the asset price as the sum of discounted future (ex tax) unit user costs; the discounting is executed by means of future interest rates.

A mathematical truth like expression (76), however, is not immediately helpful in the real world. At the beginning, or even at the end of period t , most if not all of the data that are needed for the computation of the asset prices $P_{i,j-0.5}^{t-}$ and $P_{i,j+0.5}^{t+}$ are *not* available. Thus, in practice, expression (76) must be filled in with expectations, and these depend on the point of time from which one looks at the future. A rather natural vantage point is the beginning of period t ; thus,

the operator E^{t^-} placed before a variable means that the expected value of the variable at t^- is taken. Modifying expression (76), the price at the beginning of period t of an asset of type i and age $j - 0.5$ is given by

$$(77) \quad P_{i,j-0.5}^{t^-} \equiv \frac{E^{t^-}(u_{i,j}^t - \tau_{i,j}^t)}{1 + E^{t^-} r^t} + \frac{E^{t^-}(u_{i,j+1}^{t+1} - \tau_{i,j+1}^{t+1})}{(1 + E^{t^-} r^t)(1 + E^{t^-} r^{t+1})} + \dots + \frac{E^{t^-}(u_{i,E^{t^-} J_i}^{t+E^{t^-} J_i-j} - \tau_{i,E^{t^-} J_i}^{t+E^{t^-} J_i-j})}{(1 + E^{t^-} r^t) \dots (1 + E^{t^-} r^{t+E^{t^-} J_i-j})}.$$

Notice in particular that in this expression the economically maximal age, as expected at the beginning of period t , $E^{t^-} J_i$, occurs. Put otherwise, at the beginning of period t the remaining economic lifetime of the asset is expected to be $E^{t^-} J_i - j - 0.5$ periods.²³ For each of the coming periods, there is an expected (ex tax) rental, and the (with expected interest rates) discounted rentals are summed. This sum constitutes the price (value) of the asset.

Similarly, the price at the end of period t of an asset of type i and age $j + 0.5$ is given by

$$(78) \quad P_{i,j+0.5}^{t^+} = P_{i,(j+1)-0.5}^{(t+1)^-} \equiv \frac{E^{(t+1)^-}(u_{i,j+1}^{t+1} - \tau_{i,j+1}^{t+1})}{1 + E^{(t+1)^-} r^{t+1}} + \frac{E^{(t+1)^-}(u_{i,j+2}^{t+2} - \tau_{i,j+2}^{t+2})}{(1 + E^{(t+1)^-} r^{t+1})(1 + E^{(t+1)^-} r^{t+2})} + \dots + \frac{E^{(t+1)^-}(u_{i,E^{(t+1)^-} J_i}^{t+E^{(t+1)^-} J_i-j} - \tau_{i,E^{(t+1)^-} J_i}^{t+E^{(t+1)^-} J_i-j})}{(1 + E^{(t+1)^-} r^{t+1}) \dots (1 + E^{(t+1)^-} r^{t+E^{(t+1)^-} J_i-j})}.$$

Notice that this price depends on the economically maximal age, as expected at the beginning of period $t+1$ (which is the end of period t), $E^{(t+1)^-} J_i$, which may or may not differ from the economically maximal age, as expected one period earlier, $E^{t^-} J_i$. The last mentioned expected age plays a role in the price at the end of period t of an asset of type i and age $j + 0.5$, as expected at the beginning of this period,

²³ See Erumban (2008a) on the estimation of expected lifetimes for three types of assets in a number of industries.

$$\begin{aligned}
 E^{t-} P_{i,j+0.5}^{t+} &\equiv \frac{E^{t-} (u_{i,j+1}^{t+1} - \tau_{i,j+1}^{t+1})}{1 + E^{t-} r^{t+1}} + \frac{E^{t-} (u_{i,j+2}^{t+2} - \tau_{i,j+2}^{t+2})}{(1 + E^{t-} r^{t+1})(1 + E^{t-} r^{t+2})} + \dots \\
 (79) \quad &+ \frac{E^{t-} (u_{i,E^t J_i}^{t+E^t J_i-j} - \tau_{i,E^t J_i}^{t+E^t J_i-j})}{(1 + E^{t-} r^{t+1}) \dots (1 + E^{t-} r^{t+E^t J_i-j})}.
 \end{aligned}$$

Expression (79) was obtained from expression (77) by deleting its first term as well as the first period discount factor $1 + E^{t-} r^t$. This reflects the fact that at the end of period t the asset's remaining lifetime has become shorter by one period. Generally one may expect that $E^{t-} P_{i,j+0.5}^{t+} \leq P_{i,j-0.5}^{t-}$.

Expression (78) differs from expression (79) in that expectations are at $(t+1)^-$ instead of t^- . Since one may expect that, due to technological progress, the remaining economic lifetime of any asset shortens, that is, $E^{(t+1)-} J_i \leq E^{t-} J_i$, expression (78) contains fewer terms than expression (79). Generally one may expect that $P_{i,j+0.5}^{t+} \leq E^{t-} P_{i,j+0.5}^{t+}$; that is, the actual price of an asset at the end of a period is less than or equal to the price as expected at the beginning.

Armed with these insights we return to the unit user cost expressions (68) and (69). Natural decompositions of these two expressions are

$$\begin{aligned}
 (80) \quad u_{ij}^t &= r^t P_{i,j-0.5}^{t-} + (P_{i,j-0.5}^{t-} - E^{t-} P_{i,j+0.5}^{t+}) \\
 &+ (E^{t-} P_{i,j+0.5}^{t+} - P_{i,j+0.5}^{t+}) + \tau_{ij}^t, \quad \text{for } j = 1, \dots, J_i,
 \end{aligned}$$

and

$$\begin{aligned}
 (81) \quad v_{ij}^t &= (1/2)r^t P_{ij}^t + (P_{ij}^t - E^t P_{i,j+0.5}^{t+}) \\
 &+ (E^t P_{i,j+0.5}^{t+} - P_{i,j+0.5}^{t+}) + (1/2)\tau_{ij}^t, \quad \text{for } j = 0, \dots, J_i.
 \end{aligned}$$

As before, the first term at either right-hand side represents the price of waiting. The second term, between brackets, is called anticipated time-series depreciation, and could be decomposed into the anticipated effect of time (or, anticipated revaluation) and the anticipated effect of ageing (or, anticipated cross-section depreciation). The third term, also between brackets, is called unanticipated revaluation. We will come back to these terms later.

The underlying idea is that, at the beginning of each period or, in the case of investment, at the midpoint, economic decisions are based on anticipated rather than realized prices. The fourth term in the two decompositions is again the tax term. It is here assumed that with respect to waiting and tax, anticipated and realized prices coincide.

Substituting expressions (80) and (81) into expression (70), one obtains the following aggregate decomposition,

$$\begin{aligned}
 (82) \quad C_K^t &= \sum_{i=1}^I \sum_{j=1}^{J_i} r^t P_{i,j-0.5}^{t-} K_{ij}^t + \sum_{i=1}^I \sum_{j=0}^{J_i} (1/2) r^t P_{ij}^t I_{ij}^t \\
 &+ \sum_{i=1}^I \sum_{j=1}^{J_i} (P_{i,j-0.5}^{t-} - E^{t-} P_{i,j+0.5}^{t+}) K_{ij}^t + \sum_{i=1}^I \sum_{j=0}^{J_i} (P_{i,j}^t - E^t P_{i,j+0.5}^{t+}) I_{ij}^t \\
 &+ \sum_{i=1}^I \sum_{j=1}^{J_i} (E^{t-} P_{i,j+0.5}^{t+} - P_{i,j+0.5}^{t+}) K_{ij}^t + \sum_{i=1}^I \sum_{j=0}^{J_i} (E^t P_{i,j+0.5}^{t+} - P_{i,j+0.5}^{t+}) I_{ij}^t \\
 &+ \sum_{i=1}^I \sum_{j=1}^{J_i} \tau_{ij}^t K_{ij}^t + \sum_{i=1}^I \sum_{j=0}^{J_i} (1/2) \tau_{ij}^t I_{ij}^t.
 \end{aligned}$$

On the first line after the equality sign we have the aggregate cost of waiting,

$$(83) \quad C_{K,w}^t \equiv r^t \left(\sum_{i=1}^I \sum_{j=1}^{J_i} P_{i,j-0.5}^{t-} K_{ij}^t + \sum_{i=1}^I \sum_{j=0}^{J_i} (1/2) P_{ij}^t I_{ij}^t \right).$$

Notice that the part between brackets differs slightly from expression (71). It can be interpreted as the production unit's productive capital stock *as used* during period t .

On the second line after the equality sign in expression (82) we have the aggregate cost of anticipated time-series depreciation,

$$(84) \quad C_{K,e}^t = \sum_{i=1}^I \sum_{j=1}^{J_i} (P_{i,j-0.5}^{t-} - E^{t-} P_{i,j+0.5}^{t+}) K_{ij}^t + \sum_{i=1}^I \sum_{j=0}^{J_i} (P_{i,j}^t - E^t P_{i,j+0.5}^{t+}) I_{ij}^t.$$

On the third line we have the aggregate cost of unanticipated revaluation,

$$(85) \quad C_{K,u}^t = \sum_{i=1}^I \sum_{j=1}^{J_i} (E^{t-} P_{i,j+0.5}^{t+} - P_{i,j+0.5}^{t+}) K_{ij}^t + \sum_{i=1}^I \sum_{j=0}^{J_i} (E^t P_{i,j+0.5}^{t+} - P_{i,j+0.5}^{t+}) I_{ij}^t.$$

Finally, on the fourth line we have the aggregate cost of tax,

$$(86) \quad C_{K,tax}^t = \sum_{i=1}^I \sum_{j=1}^{J_i} \tau_{ij}^t K_{ij}^t + \sum_{i=1}^I \sum_{j=0}^{J_i} (1/2) \tau_{ij}^t I_{ij}^t.$$

Using these definitions, expression (82) reduces to

$$(87) \quad C_K^t = C_{K,w}^t + C_{K,e}^t + C_{K,u}^t + C_{K,tax}^t.$$

Thus, capital input cost can rather naturally be split into four meaningful components. As will be detailed in the next section, this leads to four additional input-output models.

6. More Models

6.1 The KL-NVA model

The first two models are variants of the KL-VA model. The idea is that the (ex post) cost of time-series depreciation plus tax should be treated like the cost of intermediate inputs, and subtracted from value added. Hence, the output concept is called net value added, and defined by

$$(88) \quad NVA^t \equiv VA^t - (C_{K,e}^t + C_{K,u}^t + C_{K,tax}^t).$$

The remaining input cost is the sum of labour cost, C_L^t , and waiting cost of capital, $C_{K,w}^t$. It is assumed that $NVA^t > 0$.

Some argue that this model is to be preferred from a welfare-theoretic point of view. If the objective is to hold owned capital (including investments during the accounting period) in terms of money intact, then depreciation -- whether expected or not -- and tax should be treated like intermediate inputs (see Spant 2003). This model was strongly defended by Rymes (1983). Apart from land, he considered labour and waiting as the only primary inputs, and connected this with a Harrodian model of technological change.

The counterpart to *profitability* in this model is

$$\frac{NVA^t}{C_{K,w}^t + C_L^t},$$

and the problem is to decompose the ratios NVA^1 / NVA^0 and $(C_{K,w}^1 + C_L^1) / (C_{K,w}^0 + C_L^0)$ into price and quantity components. The decomposition of the net-value-added ratio is structurally similar to the decomposition of the value-added ratio (see appendix B). Hence, let a solution be given by

$$(89) \quad \frac{NVA^1}{NVA^0} = P_{NVA}(1,0)Q_{NVA}(1,0).$$

Using one- or two-stage Fisher indices, the input cost ratio can be decomposed as

$$(90) \quad \frac{C_{K,w}^1 + C_L^1}{C_{K,w}^0 + C_L^0} = P_{KwL}(1,0)Q_{KwL}(1,0).$$

The net-value-added based (total factor) productivity index for period 1 relative to period 0 is then defined as

$$(91) \quad IPROD_{NVA}(1,0) \equiv \frac{Q_{NVA}(1,0)}{Q_{KwL}(1,0)}.$$

In general, $IPROD_{NVA}(1,0) \neq IPROD(1,0)$. Following the reasoning of Balk (2009) it is possible to show that, if profit is zero in both periods, that is, $R^t = C^t$ ($t = 0,1$), then, for certain two-stage indices which are second-order differential approximations to Fisher indices,

$$(92) \quad \ln IPROD_{NVA}(1,0) = D'(1,0) \ln IPROD(1,0),$$

where $D'(1,0) \geq 1$ is the ratio of mean revenue over mean net value added. Since $NVA^t \leq VA^t$, it follows that $D'(1,0) \geq D(1,0)$.

The counterpart to *profit* in the KL-NVA model is $NVA^t - (C_{K,w}^t + C_L^t)$, but one easily checks that

$$(93) \quad NVA^t - (C_{K,w}^t + C_L^t) = R^t - C^t.$$

Thus, profit in the KL-NVA model is the same as profit in the KLEMS-Y model, and the same applies to their price and quantity components. Hence, there is nothing really new here.

6.2 The KL-NNVA Model

Diewert, Mizobuchi and Nomura (2005), Diewert and Lawrence (2006) and Diewert and Wykoff (2010) suggested we should consider unanticipated revaluation, which is the unanticipated part of time-series depreciation, as a monetary component that must be added to profit. The result could be called “profit from normal operations of the production unit.” Following this suggestion, the output concept becomes

$$(94) \quad NNVA^t \equiv VA^t - (C_{K,e}^t + C_{K,tax}^t),$$

which could be called normal net value added. The inputs considered are labour, C_L^t , and the waiting cost of capital, $C_{K,w}^t$. It is assumed that $NNVA^t > 0$.

The counterpart to *profitability* now is

$$\frac{NNVA^t}{C_{K,w}^t + C_L^t},$$

and the problem is to decompose the ratios $NNVA^1 / NNVA^0$ and $(C_{K,w}^1 + C_L^1) / (C_{K,w}^0 + C_L^0)$ into price and quantity components. The decomposition of the normal-net-value-added ratio is structurally similar to the decomposition of the value-added ratio (see appendix B). Hence, let a solution be given by

$$(95) \quad \frac{NNVA^1}{NNVA^0} = P_{NNVA}(1,0) Q_{NNVA}(1,0).$$

The decomposition of the input cost ratio was given by expression (90). The normal-net-value-added based (total factor) productivity index for period 1 relative to period 0 is then defined as

$$(96) \quad IPROD_{NNVA}(1,0) \equiv \frac{Q_{NNVA}(1,0)}{Q_{KwL}(1,0)}.$$

In general it will be the case that $IPROD_{NNVA}(1,0) \neq IPROD_{NVA}(1,0)$.

The counterpart to *profit* in the KL-NNVA model is $NNVA^t - (C_{K,w}^t + C_L^t)$. However, one easily checks that

$$(97) \quad NNVA^t - (C_{K,w}^t + C_L^t) = R^t - C^t + C_{K,u}^t$$

Hence, the KL-NNVA model really differs from the KLEMS-Y model.

6.3 The K-NCF model

The last two models are variants of the K-CF model. Here also the idea is that the (ex post) cost of time-series depreciation plus tax should be treated like the cost of intermediate inputs, and subtracted from cash flow. The output concept is called net cash flow, and defined by

$$(98) \quad NCF^t \equiv CF^t - (C_{K,e}^t + C_{K,u}^t + C_{K,tax}^t).$$

The remaining input cost is the waiting cost of capital, $C_{K,w}^t$. It is assumed that $NCF^t > 0$.

The counterpart to profitability now is $NCF^t / C_{K,w}^t$ and the problem is to decompose the ratios NCF^1 / NCF^0 and $C_{K,w}^1 / C_{K,w}^0$ into price and quantity components. The decomposition of the net-cash-flow ratio is structurally similar to the decomposition of the value-added ratio (see appendix B). Hence, let a solution be given by

$$(99) \quad \frac{NCF^1}{NCF^0} = P_{NCF}(1,0)Q_{NCF}(1,0).$$

Using Fisher indices, the waiting cost of capital ratio can be decomposed as

$$(100) \quad \frac{C_{K,w}^1}{C_{K,w}^0} = P_{Kw}(1,0)Q_{Kw}(1,0).$$

The net-cash-flow based (total factor) productivity index for period 1 relative to 0 is then defined as

$$(101) \quad IPROD_{NCF}(1,0) \equiv \frac{Q_{NCF}(1,0)}{Q_{Kw}(1,0)}.$$

In general, it will be the case that $IPROD_{NCF}(1,0) \neq IPROD(1,0)$. Following the reasoning of Balk (2009), it is possible to show that, if profit is zero in both periods, that is, $R^t = C^t$ ($t = 0,1$), then, for certain two-stage indices which are second-order differential approximations to Fisher indices,

$$(102) \quad \ln IPROD_{NCF}(1,0) = E'(1,0) \ln IPROD(1,0),$$

where $E'(1,0) \geq 1$ is the ratio of mean revenue over mean net cash flow. Since $NCF^t \leq CF^t$, it follows that $E'(1,0) \geq E(1,0)$.

The counterpart to *profit* in the K-NCF model is $NCF^t - C_{K,w}^t$, but one easily checks that

$$(103) \quad NCF^t - C_{K,w}^t = R^t - C^t .$$

Thus, profit in the K-NCF model is the same as profit in the KLEMS-Y model, and the same applies to their price and quantity components. Hence, there is nothing really new here.

6.4 The K-NNCF Model

A variant of the K-NCF model is obtained by considering unanticipated revaluation, which is the unanticipated part of time-series depreciation, as a component that must be added to profit. Hence, the output concept becomes

$$(104) \quad NNCF^t \equiv CF^t - (C_{K,e}^t + C_{K,tax}^t),$$

which could be called normal net cash flow. It is assumed that $NNCF^t > 0$. The only input category is the waiting cost of capital, $C_{K,w}^t$.²⁴

The counterpart to profitability now is $NNCF^t / C_{K,w}^t$, and the problem is to decompose the ratios $NNCF^1 / NNCF^0$ and $C_{K,w}^1 / C_{K,w}^0$ into price and quantity components. The decomposition of the normal-net-cash-flow ratio is structurally similar to the decomposition of the value-added ratio (see appendix B). Hence, let a solution be given by

$$(105) \quad \frac{NNCF^1}{NNCF^0} = P_{NNCF}(1,0) Q_{NNCF}(1,0) .$$

The decomposition of the input cost ratio was given by expression (100). The normal-net-value-added based (total factor) productivity index for period 1 relative to period 0 is then defined as

$$(106) \quad IPROD_{NNCF}(1,0) \equiv \frac{Q_{NNCF}(1,0)}{Q_{Kw}(1,0)} .$$

In general it will be the case that $IPROD_{NNCF}(1,0) \neq IPROD_{NCF}(1,0)$.

The counterpart to *profit* in the K-NNCF model is $NNCF^t - C_{K,w}^t$. However, one easily checks that

$$(107) \quad NNCF^t - C_{K,w}^t = R^t - C^t + C_{K,u}^t .$$

²⁴ In the model of Hulten and Schreyer (2006) total (= unanticipated plus anticipated) revaluation is added to profit. This is consistent with SNA93's prescription for non-market units.

Hence, the K-NNCF model really differs from the KLEMS-Y model. However, the K-NNCF model is the same as the KL-NNVA model, as can be concluded from a comparison of expressions (97) and (107).

7. The Rate of Return

It is useful to recall the various models in their order of appearance. We are using thereby the notation introduced gradually. Further, recall that $\Pi^t \equiv R^t - C^t$ denotes profit. The KLEMS-Y model is governed by the following accounting identity, where input categories are placed left and output categories are placed right of the equality sign:

$$(108) \quad C_{K,w}^t + C_{K,e}^t + C_{K,u}^t + C_{K,tax}^t + C_L^t + C_E^t + C_M^t + C_S^t + \Pi^t = R^t .$$

The *KL-VA* model is then seen to be governed by

$$(109) \quad C_{K,w}^t + C_{K,e}^t + C_{K,u}^t + C_{K,tax}^t + C_L^t + \Pi^t = R^t - (C_E^t + C_M^t + C_S^t) .$$

The *KL-NVA* model is governed by

$$(110) \quad C_{K,w}^t + C_L^t + \Pi^t = R^t - (C_{K,e}^t + C_{K,u}^t + C_{K,tax}^t + C_E^t + C_M^t + C_S^t) ,$$

while the *KL-NNVA* model is governed by

$$(111) \quad C_{K,w}^t + C_L^t + \Pi^{*t} = R^t - (C_{K,e}^t + C_{K,tax}^t + C_E^t + C_M^t + C_S^t) ,$$

with $\Pi^{*t} \equiv \Pi^t + C_{K,u}^t$ being the profit from normal operations. In contrast, Π^t could be called the profit from all operations.

Similarly, departing from expression (108), the *K-CF* model is seen to be governed by

$$(112) \quad C_{K,w}^t + C_{K,e}^t + C_{K,u}^t + C_{K,tax}^t + \Pi^t = R^t - (C_L^t + C_E^t + C_M^t + C_S^t) .$$

The *K-NCF* model is governed by

$$(113) \quad C_{K,w}^t + \Pi^t = R^t - (C_{K,e}^t + C_{K,u}^t + C_{K,tax}^t + C_L^t + C_E^t + C_M^t + C_S^t) ,$$

while the *K-NNCF* model is governed by

$$(114) \quad C_{K,w}^t + \Pi^{*t} = R^t - (C_{K,e}^t + C_{K,tax}^t + C_L^t + C_E^t + C_M^t + C_S^t) .$$

The last two expressions provide an excellent point of departure for a discussion of the rate of return r^t , which determines the aggregate cost of waiting or opportunity cost $C_{K,w}^t$ according to expression (83). Using definition (98) and expression (83), the accounting identity of the *K-NCF* model can be rewritten as

$$(115) \quad r^t \left(\sum_{i=1}^I \sum_{j=1}^{J_i} P_{i,j}^{t-} K_{ij}^t + \sum_{i=1}^I \sum_{j=0}^{J_i} (1/2) P_{i,j}^t I_{ij}^t \right) + \Pi^t = NCF^t .$$

Recall that the part between big brackets can be interpreted as the (value of the) production unit's capital stock as used during period t . Provided that $NCF^t \geq \Pi^t \geq 0$, the last equation then says that, apart from profit, net cash flow provides the return to the (owner of the) capital stock. This is the reason why r^t is also called the 'rate of return'.

In principle, the value of the capital stock as well as the net cash flow are empirically determined. That leaves an equation with two unknowns: the rate of return r^t and profit Π^t .

Setting $\Pi^t = 0$ and solving equation (115) for r^t delivers the so-called 'endogenous,' or 'internal' or 'balancing' rate of return. This solution is, of course, specific for the production unit. Net cash flow is calculated ex post, since it contains total time-series depreciation. Thus, the endogenous rate of return as calculated from expression (115) is also an ex post concept. The alternative is to specify some reasonable, *exogenous* value for the rate of return, say the annual percentage of headline CPI change plus something. Then, of course, profit follows from equation (115) and will in general be unequal to 0.

Thus, the endogenous rate of return is defined by the equation

$$(116) \quad r_{endo}^t \left(\sum_{i=1}^I \sum_{j=1}^{J_i} P_{i,j}^{t-} K_{ij}^t + \sum_{i=1}^I \sum_{j=1}^{J_i} (1/2) P_{ij}^t I_{ij}^t \right) = NCF^t .$$

Combining this with expression (115) delivers the following relation between the endogenous and an exogenous rate of return:

$$(117) \quad r_{endo}^t = r^t + \Pi^t / \left(\sum_{i=1}^I \sum_{j=1}^{J_i} P_{i,j}^{t-} K_{ij}^t + \sum_{i=1}^I \sum_{j=1}^{J_i} (1/2) P_{ij}^t I_{ij}^t \right) .$$

Hence, profit Π^t is positive if and only if $r_{endo}^t > r^t$.

Alternatively, using definition (104), the accounting identity of the K-NNCF model can be rewritten as

$$(118) \quad r^t \left(\sum_{i=1}^I \sum_{j=1}^{J_i} P_{i,j}^{t-} K_{ij}^t + \sum_{i=1}^I \sum_{j=0}^{J_i} (1/2) P_{i,j}^t I_{ij}^t \right) + \Pi^{*t} = NNCF^t .$$

Now, provided that $NNCF^t \geq \Pi^{*t} \geq 0$, normal net cash flow is seen as the return to the (owner of the) capital stock. Setting $\Pi^{*t} = 0$ and solving equation (118) for r^t delivers what can be called the 'normal endogenous' rate of return. In a sense, this rate absorbs not only profit but also the monetary value of all unanticipated asset revaluations. Alternatively, one can specify some reasonable, exogenous value for the rate of return. Then, of course, Π^{*t} follows from equation (118), and by subtracting the sum of all unanticipated asset revaluations, $C_{K,u}^t$, one obtains profit.

The two expressions (115) and (118) and their underlying models are polar cases. In the first, all unanticipated revaluations (that is, the whole of $C_{K,u}^t$) are considered as intermediate cost, whereas in the second they are considered as belonging to profit. Clearly, positions in between these two extremes are thinkable. For some asset types unanticipated revaluations might be considered as intermediate cost and for the remaining types these revaluations might be considered as belonging to profit.

This is a good moment to draw a number of conclusions. First, we have considered a number of input-output models: KLEMS-Y, KL-VA, KL-NVA, KL-NNVA, K-CF, K-NCF, and K-NNCF respectively. All these models lead to different (total factor) productivity *indices*. However, most of these differences are artifacts, caused by a different mixing of subtraction and division.²⁵ Thus, it depends on purpose and context of a study which particular model is chosen for the presentation of results. When productivity *indicators* are compared, the real difference turns up, namely between the KL-NNVA and K-NNCF models on the one hand and the rest on the other hand.

Second, there is no single concept of the endogenous rate of return. There is rather a continuum of possibilities, depending on the way one wants to deal with unanticipated revaluations.

Third, an endogenous rate of return, of whatever variety, can only be calculated *ex post*. Net cash flow as well as normal net cash flow require for their computation that the accounting period has expired.

Fourth, as the name suggests, a total factor productivity index or indicator suggests that all the inputs and outputs are correctly observed. Unobserved inputs and outputs and measurement errors lead to a distorted profit figure and have impacts on the interpretation of total factor productivity change. Since an endogenous rate of return can be said to absorb profit - - see expression (117) -- the extent of undercoverage has also implications for the interpretation of the rate of return (see also Schreyer 2010). Put otherwise, since an endogenous rate of return closes the gap between the input and the output side of the production unit, it is influenced by all sorts of measurement errors.

The question whether to use, for a certain production unit, an endogenous or an exogenous rate of return belongs, according to Diewert (2008), to the list of still unresolved issues. The practice of official statistical agencies is varied, as a brief survey reveals.

The U. S. Bureau of Labor Statistics uses endogenous rates (see Dean and Harper 2001),²⁶ as does Statistics Canada (see Harchaoui *et al.* 2001). The Australian Bureau of Statistics uses, per production unit considered, the maximum of the endogenous rate and a certain exogenous rate (set equal to the annual percentage change of the CPI plus 4 percent) (see Roberts 2006). Statistics New Zealand uses endogenous rates (according to their *Sources and Methods 2006* publication). The Swiss Federal Statistical office has the most intricate system: per production unit the simple mean of the endogenous rate and a certain exogenous rate is used

²⁵ Rymes (1983) would single out the KL-NVA model as the “best” one, but this is clearly not backed by the argument presented here.

²⁶ It seems to me that Jorgenson (2009) is also proposing endogenous rates of return for the four sectors considered.

as the final exogenous rate (see Rais and Sollberger 2008). Concerning the endogenous rates, however, these sources are not clear as to which concept is used precisely.

The fact that an endogenous rate of return can only be calculated ex post seems to imply that ex ante unit user costs can only be based on exogenous values for the rate of return. This, of course, implies some arbitrariness. However, since the anticipated unit user costs serve as data in economic decision processes, it is not unimportant to consider the question whether there is a sense in which such unit user costs can be based on an endogenous rate of return. This is a topic considered by Oulton (2007). The rather simple model he is using already makes clear that a fair amount of mental acrobatics is needed to combine the concept of endogeneity with that of anticipation. Let us consider the situation in our set-up.

The (at the beginning of period t) anticipated unit user cost for an asset of type i and age j over period t is, based on expression (77) and (79), given by

$$(119) \quad E^t u_{ij}^t = (E^t r^t) P_{i,j-0.5}^{t-} + (P_{i,j-0.5}^{t-} - E^t P_{i,j+0.5}^{t+}) + E^t \tau_{ij}^t \quad \text{for } j = 1, \dots, J_i.$$

These unit user costs concern assets that are available at the beginning of period t . There are, however, also investments to be made. In our set-up these investments happen at the midpoint of each period. Then, compare expression (81), the (at the midpoint of period t) anticipated unit user cost for an asset of type i and age j over the second half of period t is given by

$$(120) \quad E^t v_{ij}^t = ((1/2)E^t r^t) P_{i,j}^t + (P_{i,j}^t - E^t P_{i,j+0.5}^{t+}) + (1/2)E^t \tau_{ij}^t \quad \text{for } j = 0, \dots, J_i.$$

Anticipated total user cost over period t is now equal to

$$(121) \quad \begin{aligned} EC_K^t &\equiv (E^t r^t) \sum_{i=1}^I \sum_{j=1}^{J_i} P_{i,j-0.5}^{t-} K_{ij}^t + (E^t r^t) \sum_{i=1}^I \sum_{j=0}^{J_i} (1/2) P_{ij}^t \hat{I}_{ij}^t \\ &+ \sum_{i=1}^I \sum_{j=1}^{J_i} (P_{i,j-0.5}^{t-} - E^t P_{i,j+0.5}^{t+}) K_{ij}^t + \sum_{i=1}^I \sum_{j=0}^{J_i} (P_{i,j}^t - E^t P_{i,j+0.5}^{t+}) \hat{I}_{ij}^t \\ &+ \sum_{i=1}^I \sum_{j=1}^{J_i} (E^t \tau_{ij}^t) K_{ij}^t + \sum_{i=1}^I \sum_{j=0}^{J_i} (1/2) (E^t \tau_{ij}^t) \hat{I}_{ij}^t, \end{aligned}$$

where the quantities \hat{I}_{ij}^t ($i = 1, \dots, I; j = 0, \dots, J_i$) are as yet to be determined. Thus, given asset prices, expected asset prices, and expected amounts of tax-per-unit, expression (121) contains $\sum_{i=1}^I (1 + J_i)$ unknown investment quantities, in addition to the two rate of return terms, $E^t r^t$ and $E^t r^t$. Now this expression corresponds to the left-hand side of the accounting identity of the K-CF model. For the right-hand side we need the anticipated value of period t 's cash flow. Based on past experience, at the beginning of period t the production unit may have expectations about its output prices, and the prices of its labour, energy, materials, and services inputs. The corresponding quantities, however, are as yet to be determined. Taken together, we are having here a single equation with many unknowns and, except under heroic, simplifying assumptions, it seems difficult to get an indubitable solution for the required, endogenous rate of return.

Finally, the concept of an endogenous rate of return does not make sense for non-market units, since there is no accounting identity based on independent measures at the input and the output side.

8. Capital Utilization

Until now it was tacitly assumed that the productive capital stock was fully used in actual production. We want to make this assumption explicit. For introducing the capital utilization rate, let us return to the K-CF model, which is governed by the equation

$$(122) \quad C_K^t + \Pi^t = CF^t,$$

where C_K^t is given by expression (70). Cash flow, if positive, is seen as the return to the productive capital stock. But what if this stock is only partly used in productive operations?

Let $\{\theta_K^t K_{ij}^t, \theta_K^t I_{ij}^t; i = 1, \dots, I; j = 0, \dots, J_i\}$, where $0 < \theta_K^t < 1$, is the part of the productive capital stock that is actually used during period t . For ease of presentation the utilization rate θ_K^t is assumed to be the same for all asset types and ages. Then one easily checks that the foregoing equation, governing the K-CF model, can be written as

$$(123) \quad \theta_K^t C_K^t + (1 - \theta_K^t) C_K^t + \Pi^t = CF^t$$

where $\theta_K^t C_K^t$ and $(1 - \theta_K^t) C_K^t$ are the user costs of the used and unused parts of the capital stock, respectively.

Now, like unanticipated revaluation, one can argue that the cost of unused capital should be added to profit and that the measurement of productivity change should be based on the equation

$$(124) \quad \theta_K^t C_K^t + \Pi^{**t} = CF^t,$$

with $\Pi^{**t} \equiv (1 - \theta_K^t) C_K^t + \Pi^t$ being the profit adjusted for underutilization of capital. Put otherwise, in this model the (total factor) productivity index for period 1 relative to period 0 is defined as

$$(125) \quad IPROD_{CFU}(1,0) \equiv \frac{Q_{CF}(1,0)}{(\theta_K^1 / \theta_K^0) Q_K(1,0)}.$$

This is the index of real cash flow divided by the index of real capital input multiplied by the change of the capital utilization rate.

It is straightforward to check that the utilization rate can be introduced in any of the models discussed in this paper. This exercise is left to the reader.

9. Implementation Issues

There remain a number of implementation issues to discuss. For this, the reader is invited to return to expression (82). To ease the presentation, a period is now set equal to a year.

The quantities $\{K_{ij}^t; i = 1, \dots, I; j = 1, \dots, J_i\}$ and $\{I_{ij}^t; i = 1, \dots, I; j = 0, \dots, J_i\}$ are usually not available. Instead, as is the case in the Netherlands, the Perpetual Inventory Method generates estimates of the opening stock of assets at period $t-1$ prices $\{P_{i,j-0.5}^{t-1} K_{ij}^t = P_{i,j-0.5}^{t-1} K_{ij-0.5}^t; i = 1, \dots, I; j = 1, \dots, J_i\}$, and the Investment Survey generates estimates of mid-period values $\{P_{ij}^t I_{ij}^t; i = 1, \dots, I; j = 0, \dots, J_i\}$.

Models for time-series depreciation are briefly discussed in appendix C. The time-series depreciation for an asset of type i and age j that is available at the beginning of period t is in practice frequently modeled as

$$(126) \quad \frac{P_{i,j+0.5}^t}{P_{i,j-0.5}^t} = \frac{PPI_i^t}{PPI_i^t} (1 - \delta_{ij}),$$

where PPI_i^t denotes the Producer Price Index (or a kindred price index) that is applicable to new assets of type i , and δ_{ij} is the annual cross-section depreciation rate that is applicable to an asset of type i and age j . This depreciation rate ideally comes from an empirically estimated age-price profile.

Thus, time-series depreciation is modeled as a simple, multiplicative function of two, independent factors. The first one, PPI_i^t / PPI_i^t , which is one plus the annual rate of price change of new assets of type i , concerns the effect of the progress of time on the value of an asset of type i and age j . The second one, $1 - \delta_{ij} > 0$, concerns the effect of ageing by one year on the value of an asset of type i and age j . Ageing by one year causes the value to decline by $\delta_{ij} \times 100$ percent.

Similarly, anticipated time-series depreciation is modeled as

$$(127) \quad \frac{E^t P_{i,j+0.5}^t}{P_{i,j-0.5}^t} = E^t \left(\frac{PPI_i^t}{PPI_i^t} \right) (1 - \delta_{ij}).$$

In this expression, instead of the annual rate of price change of new assets, as observed ex post, the annual rate as expected at the beginning of period t is taken.

But what to expect? There are, of course, several options here. The first that comes to mind is to use some past, observed rate of change of PPI_i or a more general PPI . Second, one

could assume that expectedly the rate of price change of new assets is equal to the rate of change of the (headline) CPI, and use the ‘realized expectation’:

$$(128) \quad E^{t-} \left(\frac{PPI_i^{t+}}{PPI_i^{t-}} \right) = \frac{CPI^{t+}}{CPI^{t-}}.$$

Under the last assumption, the anticipated time-series depreciation is measured as

$$(129) \quad \frac{E^{t-} P_{i,j+0.5}^{t+}}{P_{i,j-0.5}^{t-}} = \frac{CPI^{t+}}{CPI^{t-}} (1 - \delta_{ij}),$$

and, combining expressions (126) and (129), unanticipated revaluation is measured by

$$(130) \quad \frac{E^{t-} P_{i,j+0.5}^{t+}}{P_{i,j-0.5}^{t-}} - \frac{P_{i,j+0.5}^{t+}}{P_{i,j-0.5}^{t-}} = \left(\frac{CPI^{t+}}{CPI^{t-}} - \frac{PPI_i^{t+}}{PPI_i^{t-}} \right) (1 - \delta_{ij}).$$

Similar expressions hold for assets that are acquired at the midpoint of period t , except that we must make a distinction between new and used assets. The time-series depreciation for an asset of type i and age j is modeled as

$$(131) \quad \begin{aligned} \frac{P_{i,0.5}^{t+}}{P_{i,0}^t} &= \frac{PPI_i^{t+}}{PPI_i^t} (1 - \delta_{i0}) \\ \frac{P_{i,j+0.5}^{t+}}{P_{i,j}^t} &= \frac{PPI_i^{t+}}{PPI_i^t} (1 - \delta_{ij} / 2), \end{aligned} \quad \text{for } j = 1, \dots, J_i.$$

The anticipated time-series depreciation is measured by

$$(132) \quad \begin{aligned} \frac{E^t P_{i,0.5}^{t+}}{P_{i,0}^t} &= \frac{CPI^{t+}}{CPI^t} (1 - \delta_{i0}) \\ \frac{E^t P_{i,j+0.5}^{t+}}{P_{i,j}^t} &= \frac{CPI^{t+}}{CPI^t} (1 - \delta_{ij} / 2), \end{aligned} \quad \text{for } j = 1, \dots, J_i,$$

and unanticipated revaluation is measured by

$$(133) \quad \frac{E^t P_{i,0.5}^{t+}}{P_{i,0}^t} - \frac{P_{i,0.5}^{t+}}{P_{i,0}^t} = \left(\frac{CPI^{t+}}{CPI^t} - \frac{PPI_i^{t+}}{PPI_i^t} \right) (1 - \delta_{i0})$$

$$\frac{E^t P_{i,j+0.5}^{t+}}{P_{i,j}^t} - \frac{P_{i,j+0.5}^{t+}}{P_{i,j}^t} = \left(\frac{CPI^{t+}}{CPI^t} - \frac{PPI_i^{t+}}{PPI_i^t} \right) (1 - \delta_{ij} / 2) \quad \text{for } j = 1, \dots, J_i.$$

An important question is in which circumstances the unit user costs u_{ij}^t and v_{ij}^t become non-positive? Consider, for instance, expression (80), and substitute expressions (129) and (130). This yields

$$(134) \quad \frac{u_{ij}^t}{P_{i,j-0.5}^{t-}} = r^t + 1 - \frac{CPI^{t+}}{CPI^{t-}} (1 - \delta_{ij}) + \left(\frac{CPI^{t+}}{CPI^{t-}} - \frac{PPI_i^{t+}}{PPI_i^{t-}} \right) (1 - \delta_{ij}) + \frac{\tau_{ij}^t}{P_{i,j-0.5}^{t-}}$$

$$= r^t + 1 - \frac{PPI_i^{t+}}{PPI_i^{t-}} (1 - \delta_{ij}) + \frac{\tau_{ij}^t}{P_{i,j-0.5}^{t-}}.$$

Hence, $u_{ij}^t \leq 0$ if and only if

$$(135) \quad \frac{PPI_i^{t+}}{PPI_i^{t-}} \geq \frac{1 + r^t + \tau_{ij}^t / P_{i,j-0.5}^{t-}}{1 - \delta_{ij}}.$$

In certain, extreme cases this can indeed happen. Consider assets with a very low cross-sectional depreciation rate (such as certain buildings or land) and a very high revaluation rate (or rate of price increase). A low interest plus tax rate can then lead to negative unit user costs. Put otherwise, when the ex post revaluation (as measured by a PPI) more than offsets interest plus tax plus depreciation then the unit user cost of such an asset becomes negative.

If the unanticipated revaluation is deleted from the user cost, that is, unit user cost is measured by

$$(136) \quad \frac{u_{ij}^t}{P_{i,j-0.5}^{t-}} = r^t + 1 - \frac{CPI^{t+}}{CPI^{t-}} (1 - \delta_{ij}) + \frac{\tau_{ij}^t}{P_{i,j-0.5}^{t-}},$$

then $u_{ij}^t \leq 0$ if and only if

$$(137) \quad \frac{CPI^{t^+}}{CPI^{t^-}} \geq \frac{1 + r^t + \tau_{ij}^t / P_{i,j}^{t^-0.5}}{1 - \delta_{ij}}.$$

The likelihood that such a situation will occur is small. For this to happen, expected revaluation (as measured by a *CPI*) must more than offset interest plus tax plus depreciation.

10. The Netherlands' System in Perspective

Against the backdrop of the preceding analysis, I now briefly review the Netherlands' system of productivity statistics, as laid out in Van den Bergen *et al.* (2007). Basically the system is built on the KLEMS-Y and KL-VA models.

Revenue R (or the value of gross output), value added VA , and intermediate inputs cost C_{EMS} is obtained from National Accounts' supply and use tables at current and previous year prices. The level of detail is a cross classification of 120 industries and 275 commodity groups. When it comes to consolidation, imputations must be made for trade and transport margins. The reason is that inter-industry deliveries of these margins are not recorded, but must be estimated from column and row totals.

The quantity indices $Q_R(t, t-1)$, $Q_{VA}(t, t-1)$, and $Q_{EMS}(t, t-1)$ are, for the time being and to be consistent with National Accounts' practice, chosen as Laspeyres.

Labour cost, C_L , is based on a cross-classification of two types (employees and self-employed workers) and 49 industries. The unit of measurement is an hour worked. It is assumed that, with some exceptions, in each industry self-employed workers have the same annual income as employees. Again, the quantity index $Q_L(t, t-1)$ is Laspeyres.

The cost of capital input, C_K , is based on a cross-classification of 20 asset types by 60 industries by 18 institutional sectors. Beginning of year estimates of the available capital stock are generated by a version of the Perpetual Inventory Method, whereas the annual Investment Survey delivers the values of additions to and subtractions from the capital stock. User cost is calculated according to expression (70), with (68) and (69) substituted, except that at the level of asset type (and age) the tax (plus subsidies) components are not known. Thus, the tax (plus subsidies) components must be inserted at a higher level of aggregation. Wherever necessary, beginning and end of year price index numbers are approximated by geometric means of adjacent year (average) annual price index numbers. For instance, $PPI_i^{t^+}$ is approximated by $(PPI_i^t PPI_i^{t+1})^{1/2}$. The quantity index $Q_K(t, t-1)$ is Laspeyres. All the operational details are discussed by Balk and Van den Bergen (2006). The capital utilization rate is set equal to 1.

The interest rate r^t is set equal to the so-called Internal Reference Rate, which is the interest rate that banks charge to each other, plus 1.5 percent. For all the assets, unanticipated revaluation is retained as part of their unit user cost.²⁷

Tying the various strands together, the gross output based total factor productivity index is computed as

$$(138) \quad IPROD(t, t-1) = \frac{Q_R(t, t-1)}{\left[\frac{C_K^{t-1} Q_K(t, t-1) + C_L^{t-1} Q_L(t, t-1) + C_{EMS}^{t-1} Q_{EMS}(t, t-1)}{C_K^{t-1} + C_L^{t-1} + C_{EMS}^{t-1}} \right]}$$

and the value-added based total factor productivity index as

$$(139) \quad IPROD_{VA}(t, t-1) = \frac{Q_{VA}(t, t-1)}{\left[\frac{C_K^{t-1} Q_K(t, t-1) + C_L^{t-1} Q_L(t, t-1)}{C_K^{t-1} + C_L^{t-1}} \right]}$$

A number of sensitivity analyses were performed to gauge the influence of assumptions on outcomes. I review the main results:

1. Changing the index formula, from Laspeyres to Paasche and Fisher, did not lead to remarkable adjustments.
2. For all the assets, unanticipated revaluation was excluded from their unit user cost, which means that it was added to profit. This led to small, immaterial differences between the TFP index numbers.
3. Varying the exogenous interest rate, for instance by setting it equal to the annual rate of change of the headline CPI plus 4 percent, also caused relatively small changes.
4. Using endogenous interest rates, computed according to expression (115), had considerably more impact. The endogenous rates themselves showed a substantial variability, both cross-sectionally (over industries) and intertemporally. Moreover, there appeared to be a strong dependence on the imputation method used for the compensation of self-employed workers. The resulting TFP index numbers varied wildly, especially in agriculture and the mining industry.²⁸

Interesting empirical results were obtained by Vancauteren *et al.* (2009). Over the years 1995 to 2007 these authors calculated total factor productivity changes according to the KLEMS-Y, KL-

²⁷ The only exception, by SNA93 conventions motivated, concerns the asset type “transfer of property rights.”

²⁸ It is also interesting to look at the results of the sensitivity analyses carried out by Erumban (2008b), in particular those concerning the interest rate and the revaluation part of the user cost. MacGibbon (2008) compared results from endogenous and exogenous rates of return and from the in- and exclusion of revaluations. Also interesting are the numerical results obtained by Coremberg (2008). He compared results from using Laspeyres and Törnqvist indices, undifferentiated and differentiated labour input categories, and capital input quantity indices for stocks and services. The biggest effect came from the introduction of a capital utilization rate.

VA, KL-NVA, K-CF and K-NCF models, with exogenous and endogenous interest rates, for nine industrial sectors and their aggregate.

11. Conclusion

After measurement comes explanation. Depending on the initial level of aggregation, there appear to be two main directions. The first is disaggregation: the explanation of productivity change at an aggregate level (economy, sector, industry) by productivity change at a lower level (firm, plant) and other factors, collectively subsumed under the heading of re-allocation (expansion, contraction, entry, and exit of units). This topic was reviewed by Balk (2003, section 6). As the example of Balk and Hoogenboom-Spijker (2003) demonstrates, this type of research is of an economic-statistical nature, and there are no neoclassical assumptions involved.

The second direction is concerned with the decomposition of productivity change into factors such as technological change, technical efficiency change, scale effects, input- and output-mix effects, and chance. The basic idea can be explained as follows.

To start with, for each time period t the technology to which the production unit under consideration has access is defined as the set S^t of all the input-output quantity combinations which are feasible during t . Such a set is assumed to have nice properties like being closed, bounded, and convex. Of particular interest is the subset of S^t , called its frontier, consisting of all the efficient input-output combinations. An input-output quantity combination is called efficient when output cannot be increased without increasing some input and input cannot be decreased without decreasing some output.

From base period to comparison period our production unit moves from $(x^0, y^0) \in S^0$ to $(x^1, y^1) \in S^1$. Decomposition of productivity change means that between these two points some hypothetical path is constructed, the segments of which can be given a distinct interpretation.

In particular, we consider the projection of (x^0, y^0) on the frontier of S^0 , and the projection of (x^1, y^1) on the frontier of S^1 . Comparing the base period and comparison period distances between the original points and their projections provides a measure of efficiency change.

Two more points are given by projecting (x^0, y^0) also on the frontier of S^1 , and (x^1, y^1) on the frontier of S^0 . The distance between the two frontiers at the base and comparison period projection points provides a (local) measure of technological change. And, finally, moving over each frontier (which is a surface in $N + M$ -dimensional space) from a base period to a comparison period projection point provides measures of the scale and input-output mix effects.

The construction of all those measures was discussed by Balk (2004). Since there is no unique path connecting the two observations, there is no unique decomposition either.

And here come the neoclassical assumptions, at the end of the day rather than at its beginning. Suppose that the production unit always stays on the frontier, that its input- and output-mix is optimal at the, supposedly given, input and output prices, and that the two technology sets exhibit constant returns to scale, then productivity change reduces to technological change (see Balk (1998, section 3.7) for a formal proof). The technology sets are thereby supposed to reflect the true state of nature, which rules out chance as a factor also contributing to productivity change.²⁹

Appendix A. Indices and Indicators

The basic measurement tools used are price and quantity indices and indicators. The first are ratio-type measures, and the second are difference-type measures. What, precisely, are the requirements for good tools?

A1. Indices

A *price index* is a positive, continuously differentiable function $P(p^1, y^1, p^0, y^0): \mathfrak{R}_{++}^{4N} \rightarrow \mathfrak{R}_{++}$ that correctly indicates any increase or decrease of the elements of the price vectors p^1 or p^0 , conditional on the quantity vectors y^1 and y^0 . A *quantity index* is a positive, continuously differentiable function of the same variables $Q(p^1, y^1, p^0, y^0): \mathfrak{R}_{++}^{4N} \rightarrow \mathfrak{R}_{++}$ that correctly indicates any increase or decrease of the elements of the quantity vectors y^1 and y^0 , conditional on the price vectors p^1 or p^0 . The number N is called the dimension of the price or quantity index.

The basic requirements on price and quantity indices are:

A1. Monotonicity in prices. $P(p^1, y^1, p^0, y^0)$ is increasing in comparison period prices p_n^1 and decreasing in base period prices p_n^0 ($n = 1, \dots, N$).

A1'. Monotonicity in quantities. $Q(p^1, y^1, p^0, y^0)$ is increasing in comparison period quantities y_n^1 and decreasing in base period quantities y_n^0 ($n = 1, \dots, N$).

A2. Linear homogeneity in comparison period prices. Multiplication of all comparison period prices by a common factor leads to multiplication of the price index number by this factor; that is, $P(\lambda p^1, y^1, p^0, y^0) = \lambda P(p^1, y^1, p^0, y^0)$ ($\lambda > 0$).

A2'. Linear homogeneity in comparison period quantities. Multiplication of all comparison period quantities by a common factor leads to multiplication of the quantity index number by this factor; that is, $Q(p^1, \lambda y^1, p^0, y^0) = \lambda Q(p^1, y^1, p^0, y^0)$ ($\lambda > 0$).

²⁹ On stochastic productivity measurement see Chambers (2008).

A3. Identity test. If all the comparison period prices are equal to the corresponding base period prices, then the price index number must be equal to 1: $P(p^1, y^1, p^0, y^0) = 1$.

A3'. Identity test. If all the comparison period quantities are equal to the corresponding base period quantities, then the quantity index number must be equal to 1: $Q(p^0, y^0, p^0, y^0) = 1$.

A4. Homogeneity of degree 0 in prices. Multiplication of all comparison and base period prices by the same factor does not change the price index number; that is, $P(\lambda p^1, y^1, \lambda p^0, y^0) = P(p^1, y^1, p^0, y^0)$ ($\lambda > 0$).

A4'. Homogeneity of degree 0 in quantities. Multiplication of all comparison period and base period quantities by the same factor does not change the quantity index number; that is, $Q(p^1, \lambda y^1, p^0, \lambda y^0) = Q(p^1, y^1, p^0, y^0)$ ($\lambda > 0$).

A5. Dimensional invariance. The price index is invariant to changes in the units of measurement of the commodities: for any diagonal matrix Λ with elements of \mathfrak{R}_{++} it is required that $P(p^1 \Lambda, y^1 \Lambda^{-1}, p^0 \Lambda, y^0 \Lambda^{-1}) = P(p^1, y^1, p^0, y^0)$.

A5'. Dimensional invariance. The quantity index is invariant to changes in the units of measurement of the commodities: for any diagonal matrix Λ with elements of \mathfrak{R}_{++} , it is required that $Q(p^1 \Lambda, y^1 \Lambda^{-1}, p^0 \Lambda, y^0 \Lambda^{-1}) = Q(p^1, y^1, p^0, y^0)$.

Product Test. $P(p^1, y^1, p^0, y^0)Q(p^1, y^1, p^0, y^0) = p^1 \cdot y^1 / p^0 \cdot y^0$.

Any function $P(p^1, y^1, p^0, y^0)$ that satisfies axiom A5 can be written as a function of only $3N$ variables, namely the price relatives p_n^1 / p_n^0 , the comparison period values $v_n^1 \equiv p_n^1 y_n^1$, and the base period values $v_n^0 \equiv p_n^0 y_n^0$ ($n = 1, \dots, N$).

Similarly, any function $Q(p^1, y^1, p^0, y^0)$ that satisfies axiom A5' can be written as a function of only $3N$ variables, namely the quantity relatives y_n^1 / y_n^0 , the comparison period values $v_n^1 \equiv p_n^1 y_n^1$, and the base period values $v_n^0 \equiv p_n^0 y_n^0$ ($n = 1, \dots, N$).

Some simple examples might be useful to illustrate this. Consider the Laspeyres price index as a function of prices and quantities,

$$P^L(p^1, y^1, p^0, y^0) \equiv p^1 \cdot y^0 / p^0 \cdot y^0,$$

and notice that this index can be written as a function of price relatives and (base period) values,

$$P^L(p^1, y^1, p^0, y^0) = \sum_{n=1}^N (p_n^1 / p_n^0) v_n^0 / \sum_{n=1}^N v_n^0.$$

Similarly, the Paasche price index

$$P^P(p^1, y^1, p^0, y^0) \equiv p^1 \cdot y^1 / p^0 \cdot y^1,$$

can be written as a function of price relatives and (comparison period) values,

$$P^P(p^1, y^1, p^0, y^0) = \left(\sum_{n=1}^N (p_n^0 / p_n^1) v_n^1 / \sum_{n=1}^N v_n^1 \right)^{-1}.$$

Finally, the Fisher price index, defined as the geometric mean of the Laspeyres and Paasche indices, reads

$$P^F(p^1, y^1, p^0, y^0) \equiv \left[\frac{\sum_{n=1}^N (p_n^1 / p_n^0) v_n^0 / \sum_{n=1}^N v_n^0}{\sum_{n=1}^N (p_n^0 / p_n^1) v_n^1 / \sum_{n=1}^N v_n^1} \right]^{1/2}.$$

Such functional forms are useful for the definition of two-stage indices. Let the aggregate under consideration be denoted by A , and let A be partitioned arbitrarily into K subaggregates A_k ,

$$A = \cup_{k=1}^K A_k, \quad A_k \cap A_{k'} = \phi, \quad \text{for } k \neq k'.$$

Each subaggregate consists of a number of items. Let $N_k \geq 1$ denote the number of items contained in A_k ($k = 1, \dots, K$). Obviously $N = \sum_{k=1}^K N_k$. Let $(p_k^1, y_k^1, p_k^0, y_k^0)$ be the subvector of (p^1, y^1, p^0, y^0) corresponding to the subaggregate A_k . Recall that $v_n^t \equiv p_n^t y_n^t$ is the value of item n at period t . Then $V_k^t \equiv \sum_{n \in A_k} v_n^t$ ($k = 1, \dots, K$) is the value of the subaggregate A_k at period t , and $V^t \equiv \sum_{n \in A} v_n^t = \sum_{k=1}^K V_k^t$ is the value of aggregate A at period t .

Let $P(\cdot)$, $P^{(1)}(\cdot)$, ..., $P^{(K)}(\cdot)$ be price indices of dimension K, N_1, \dots, N_K respectively that satisfy A1, ..., A5. Then the price index defined by

$$(140) \quad P^*(p^1, y^1, p^0, y^0) \equiv P(P^{(k)}(p_k^1, x_k^1, p_k^0, x_k^0), V_k^1, V_k^0; k = 1, \dots, K)$$

is of dimension N and also satisfies A1, ..., A5. The index $P^*(\cdot)$ is called a two-stage index. The first stage refers to the indices $P^{(k)}(\cdot)$ for the subaggregates A_k ($k = 1, \dots, K$). The second stage refers to the index $P(\cdot)$ that is applied to the subindices $P^{(k)}(\cdot)$ ($k = 1, \dots, K$). A two-stage index such as is defined by expression (140) closely corresponds to the calculation practice at statistical agencies. All the subindices are usually of the same functional form, for instance Laspeyres or Paasche indices. The aggregate, second stage index may or may not be of the same functional form. This could be, for instance, a Fisher index.

If the functional forms of the subindices $P^{(k)}(\cdot)$ ($k = 1, \dots, K$) and the aggregate index $P(\cdot)$ are the same, then $P^*(\cdot)$ is called a two-stage $P(\cdot)$ index. Continuing the example, the two-stage Laspeyres price index reads

$$P^{*L}(p^1, y^1, p^0, y^0) \equiv \sum_{k=1}^K P^L(p_k^1, y_k^1, p_k^0, y_k^0) V_k^0 / \sum_{k=1}^K V_k^0,$$

and one simply checks that the two-stage and the single-stage Laspeyres price indices coincide. However, this is the exception rather than the rule. For most indices, two-stage and single-stage variants do not coincide.

Similarly, let $Q(\cdot)$, $Q^{(1)}(\cdot)$, ..., $Q^{(K)}(\cdot)$ be quantity indices of dimension K, N_1, \dots, N_K respectively that satisfy $A1', \dots, A5'$. Then the quantity index defined by

$$(141) \quad Q^*(p^1, y^1, p^0, y^0) \equiv Q(Q^{(k)}(p_k^1, y_k^1, p_k^0, y_k^0), V_k^1, V_k^0; k = 1, \dots, K)$$

is of dimension N and also satisfies $A1', \dots, A5'$. The index $Q^*(\cdot)$ is called a two-stage index.

A2. Indicators

Provided that certain reasonable requirements are satisfied, the continuous functions $P(p^1, y^1, p^0, y^0): \mathfrak{R}_{++}^{4N} \rightarrow \mathfrak{R}$ and $Q(p^1, y^1, p^0, y^0): \mathfrak{R}_{++}^{4N} \rightarrow \mathfrak{R}$ will be called a price indicator and a quantity indicator respectively. Notice that these functions may take on negative or zero values. The basic requirements are:

AA1. Monotonicity in prices. $P(p^1, y^1, p^0, y^0)$ is increasing in comparison period prices p_n^1 and decreasing in base period prices p_n^0 ($n = 1, \dots, N$).

AA1'. Monotonicity in quantities. $Q(p^1, y^1, p^0, y^0)$ is increasing in comparison period quantities y_n^1 and decreasing in base period quantities y_n^0 ($n = 1, \dots, N$).

AA3. Identity test. If all the comparison period prices are equal to the corresponding base period prices, then the price indicator must deliver the outcome 0: $P(p^0, y^1, p^0, y^0) = 0$.

AA3'. Identity test. If all the comparison period quantities are equal to the corresponding base period quantities, then the quantity indicator must deliver the outcome 0: $Q(p^1, y^0, p^0, y^0) = 0$.

AA4. Homogeneity of degree 1 in prices. Multiplication of all comparison and base period prices by a common factor changes the price indicator outcome by this factor; that is, $P(\lambda p^1, y^1, \lambda p^0, y^0) = \lambda P(p^1, y^1, p^0, y^0)$ ($\lambda > 0$).

AA4'. Homogeneity of degree 1 in quantities. Multiplication of all comparison period and base period quantities by a common factor changes the quantity indicator outcome by this factor; that is, $Q(p^1, \lambda y^1, p^0, \lambda y^0) = \lambda Q(p^1, y^1, p^0, y^0)$ ($\lambda > 0$).

AA5. Dimensional invariance. The price indicator is invariant to changes in the units of measurement of the commodities: for any diagonal matrix Λ with elements of \mathfrak{R}_{++} , it is required that $P(p^1 \Lambda, y^1 \Lambda^{-1}, p^0 \Lambda, y^0 \Lambda^{-1}) = P(p^1, y^1, p^0, y^0)$.

AA5' . Dimensional invariance. The quantity indicator is invariant to changes in the units of measurement of the commodities: for any diagonal matrix Λ with elements of \mathfrak{R}_{++} , it is required that $Q(p^1\Lambda, y^1\Lambda^{-1}, p^0\Lambda, y^0\Lambda^{-1}) = Q(p^1, y^1, p^0, y^0)$.

Analogue of the Product Test. $P(p^1, y^1, p^0, y^0) + Q(p^1, y^1, p^0, y^0) = p^1 \cdot y^1 - p^0 \cdot y^0$.

Any function $P(p^1, y^1, p^0, y^0)$ that satisfies axiom AA5 can be written as a function of only $3N$ variables, namely the price relatives p_n^1 / p_n^0 , the comparison period values $v_n^1 \equiv p_n^1 y_n^1$, and the base period values $v_n^0 \equiv p_n^0 y_n^0$ ($n = 1, \dots, N$).

Similarly, any function $Q(p^1, y^1, p^0, y^0)$ that satisfies axiom AA5' can be written as a function of only $3N$ variables, namely the quantity relatives y_n^1 / y_n^0 , the comparison period values $v_n^1 \equiv p_n^1 y_n^1$, and the base period values $v_n^0 \equiv p_n^0 y_n^0$ ($n = 1, \dots, N$).

Also here some simple examples might be useful. Consider the Laspeyres price indicator as a function of prices and quantities,

$$P^L(p^1, y^1, p^0, y^0) \equiv (p^1 - p^0) \cdot y^0,$$

and notice that this indicator can be written as a function of price relatives and (base period) values,

$$P^L(p^1, y^1, p^0, y^0) \equiv \sum_{n=1}^N (p_n^1 / p_n^0 - 1)v_n^0.$$

Similarly, the Paasche price indicator,

$$P^P(p^1, y^1, p^0, y^0) \equiv (p^1 - p^0) \cdot y^1,$$

can be written as a function of price relatives and (comparison period) values,

$$P^P(p^1, y^1, p^0, y^0) \equiv \sum_{n=1}^N (1 - p_n^0 / p_n^1)v_n^1.$$

Finally, the Bennet indicator is usually defined as

$$P^B(p^1, y^1, p^0, y^0) \equiv (1/2)(p^1 - p^0) \cdot (y^1 + y^0),$$

but can be written as

$$P^B(p^1, y^1, p^0, y^0) = (1/2) \left[\sum_{n=1}^N (p_n^1 / p_n^0 - 1)v_n^0 + \sum_{n=1}^N (1 - p_n^0 / p_n^1)v_n^1 \right].$$

The Bennet price indicator for an aggregate is a simple sum of Bennet price indicators for its subaggregates:

$$P^B(p^1, y^1, p^0, y^0) = \sum_{k=1}^K P^B(p_k^1, y_k^1, p_k^0, y_k^0),$$

and a similar relation holds for quantity indicators.

Appendix B. Decompositions of the Value Added Ratio

Value added is defined as revenue minus the cost of intermediate inputs. For the logarithm of the value added ratio, we get by repeated application of the logarithmic mean³⁰ $L(a, b)$,

$$\begin{aligned}
 \ln\left(\frac{VA^1}{VA^0}\right) &= \frac{VA^1 - VA^0}{L(VA^1, VA^0)} \\
 (142) \quad &= \frac{R^1 - R^0}{L(VA^1, VA^0)} - \frac{C_{EMS}^1 - C_{EMS}^0}{L(VA^1, VA^0)} \\
 &= \frac{L(R^1, R^0) \ln(R^1 / R^0)}{L(VA^1, VA^0)} - \frac{L(C_{EMS}^1, C_{EMS}^0) \ln(C_{EMS}^1 / C_{EMS}^0)}{L(VA^1, VA^0)}.
 \end{aligned}$$

For R^1 / R^0 recall expression (7) and decompose the ratio C_{EMS}^1 / C_{EMS}^0 by one- or two stage Fisher indices as

$$\begin{aligned}
 (143) \quad C_{EMS}^1 / C_{EMS}^0 &= P^F(w_{EMS}^1, x_{EMS}^1, w_{EMS}^0, x_{EMS}^0) Q^F(w_{EMS}^1, x_{EMS}^1, w_{EMS}^0, x_{EMS}^0) \\
 &\equiv P_{EMS}(1,0) Q_{EMS}(1,0).
 \end{aligned}$$

Then the logarithm of the value added ratio can be expressed as

$$\begin{aligned}
 (144) \quad \ln\left(\frac{VA^1}{VA^0}\right) &= \frac{L(R^1, R^0) \ln(P_R(1,0) Q_R(1,0))}{L(VA^1, VA^0)} \\
 &\quad - \frac{L(C_{EMS}^1, C_{EMS}^0) \ln(P_{EMS}(1,0) Q_{EMS}(1,0))}{L(VA^1, VA^0)}.
 \end{aligned}$$

This can be rearranged as

$$(145) \quad \frac{VA^1}{VA^0} = \frac{P_R(1,0)^\phi}{P_{EMS}(1,0)^\psi} \frac{Q_R(1,0)^\phi}{Q_{EMS}(1,0)^\psi},$$

where $\phi \equiv L(R^1, R^0) / L(VA^1, VA^0)$, that is, as mean revenue over mean value added, and $\psi \equiv L(C_{EMS}^1, C_{EMS}^0) / L(VA^1, VA^0)$, that is, as mean intermediate inputs cost over mean value added. Thus, value added price and quantity indices can rather naturally be defined by

$$(146) \quad P_{VA}(1,0) \equiv \frac{P_R(1,0)^\phi}{P_{EMS}(1,0)^\psi},$$

³⁰ For any two strictly positive real numbers a and b their logarithmic mean is defined by $L(a, b) = (a - b) / \ln(a / b)$ if $a \neq b$ and $L(a, a) = a$.

$$(147) \quad Q_{VA}(1,0) \equiv \frac{Q_R(1,0)^\phi}{Q_{EMS}(1,0)^\psi}.$$

These indices generalize the conventional Montgomery-Vartia indices (see Balk 2008, 87 for their definition). They are Consistent-in-Aggregation, but fail the Equality Test. The reason is that,

$$(148) \quad \phi - \psi = \frac{L(R^1, R^0) - L(C_{EMS}^1, C_{EMS}^0)}{L(VA^1, VA^0)} \leq 1,$$

because $L(a,1)$ is a concave function.

An alternative is to define $P_{VA}(1,0)$ as a Fisher-type index of the subindices $P_R(1,0)$ and $P_{EMS}(1,0)$; that is,

$$(149) \quad P_{VA}(1,0) \equiv \left[\frac{\frac{R^0}{VA^0} P_R(1,0) - \frac{C_{EMS}^0}{VA^0} P_{EMS}(1,0)}{\frac{R^1}{VA^1} (P_R(1,0))^{-1} - \frac{C_{EMS}^1}{VA^1} (P_{EMS}(1,0))^{-1}} \right]^{1/2}.$$

The numerator is a Laspeyres-type double deflator, and the denominator is the inverse of a Paasche-type double deflator. Similarly, $Q_{VA}(1,0)$ is defined as a Fisher-type index of the subindices $Q_R(1,0)$ and $Q_{EMS}(1,0)$. These indices satisfy the Equality Test, but fail the Consistency-in-Aggregation Test. Moreover, there are situations where Fisher-type indices are undefined.

Appendix C. Decompositions of Time Series Depreciation

Time-series depreciation of an asset of type i and age j over period t is, according to expression (68), defined by $P_{i,j-0.5}^{t^-} - P_{i,j+0.5}^{t^+}$, which is the (nominal) value change of the asset between the beginning and the end of the period. This value change combines the effect of the progress of time, from t^- to t^+ , with the effect of ageing, from $j-0.5$ to $j+0.5$. Since value change is here measured as a difference, a natural decomposition of time-series depreciation according to these two effects is

$$(150) \quad P_{i,j-0.5}^{t^-} - P_{i,j+0.5}^{t^+} = (1/2)[(P_{i,j-0.5}^{t^-} - P_{i,j-0.5}^{t^+}) + (P_{i,j+0.5}^{t^-} - P_{i,j+0.5}^{t^+})] \\ + (1/2)[(P_{i,j-0.5}^{t^-} - P_{i,j+0.5}^{t^-}) + (P_{i,j-0.5}^{t^+} - P_{i,j+0.5}^{t^+})].$$

This decomposition is symmetric. The first term on the right-hand side of the equality sign measures the effect of the progress of time on an asset of unchanged age; this is called revaluation. The revaluation, as measured here, is the arithmetic mean of the revaluation of a

$j - 0.5$ periods old asset and a $j + 0.5$ periods old asset, and may be said to hold for a j periods old asset.

The second term concerns the effect of ageing, which is measured by the price difference of two, otherwise identical, assets that differ precisely one period in age. This is called Hicksian or cross-section depreciation. The arithmetic mean is taken of cross-section depreciation at the beginning and end of the period, and, hence, may be said to hold at the midpoint of period t .

Since the Perpetual Inventory Method combines the beginning-of-period price with the corresponding cohort quantities, expression (150) is rewritten as

$$(151) \quad 1 - \frac{P_{i,j+0.5}^{t+}}{P_{i,j-0.5}^{t-}} = \frac{(1/2)[(P_{i,j-0.5}^{t-} - P_{i,j-0.5}^{t+}) + (P_{i,j+0.5}^{t-} - P_{i,j+0.5}^{t+})]}{P_{i,j-0.5}^{t-}} + \frac{(1/2)[(P_{i,j-0.5}^{t-} - P_{i,j+0.5}^{t-}) + (P_{i,j-0.5}^{t+} - P_{i,j+0.5}^{t+})]}{P_{i,j-0.5}^{t-}}.$$

At the left-hand side of this expression we have $(P_{i,j+0.5}^{t+} / P_{i,j-0.5}^{t-})$ as an inverse ratio-type measure of time-series depreciation. Considered as a decomposition, however, expression (151) is not symmetric. A symmetrical decomposition is given by

$$(152) \quad \frac{P_{i,j+0.5}^{t+}}{P_{i,j-0.5}^{t-}} = \left[\frac{P_{i,j-0.5}^{t+}}{P_{i,j-0.5}^{t-}} \frac{P_{i,j+0.5}^{t+}}{P_{i,j+0.5}^{t-}} \right]^{1/2} \left[\frac{P_{i,j+0.5}^{t+}}{P_{i,j-0.5}^{t-}} \frac{P_{i,j+0.5}^{t-}}{P_{i,j-0.5}^{t-}} \right]^{1/2}.$$

The first term on the right-hand side of the equality sign measures revaluation. The second term measures cross-section depreciation. As one can see, revaluation depends on age, and cross-section depreciation depends on time. In the usual model, these two dependencies are assumed away. Revaluation is approximated by P_i^{t+} / P_i^{t-} , the price change of a *new* asset of type i from beginning to end of period t . Cross-section depreciation is approximated by $1 - \delta_{ij}$, where δ_{ij} is the percentage of annual depreciation that applies to an asset of type i and age j . The specific formulation highlights the fact that ageing usually diminishes the value of an asset.

Under these two assumptions, the basic time-series depreciation model for an asset of type i and age j , over period t , is given by

$$(153) \quad \frac{P_{i,j+0.5}^{t+}}{P_{i,j-0.5}^{t-}} = \frac{P_i^{t+}}{P_i^{t-}} (1 - \delta_{ij}) \quad \text{for } j = 1, \dots, J_i.$$

For assets that are acquired at the midpoint of period t one must distinguish between new and used assets. Over the second half of period t , the model reads

$$(154) \quad \frac{P_{i,0.5}^{t+}}{P_{i,0}^t} = \frac{P_i^{t+}}{P_i^t} (1 - \delta_{i0})$$

$$\frac{P_{i,j+0.5}^{t+}}{P_{i,j}^t} = \frac{P_i^{t+}}{P_i^t} (1 - \delta_{ij} / 2) \quad \text{for } j = 1, \dots, J_i,$$

where $(1 - \delta_{ij} / 2)$ serves as an approximation to $(1 - \delta_{ij})^{1/2}$. The percentage of annual depreciation, δ_{ij} , ideally comes from an empirically estimated age-price profile for asset-type i . Under a geometric profile one specifies $\delta_{i0} = \delta_i / 2$ and $\delta_{ij} = \delta_i$ ($j = 1, \dots, J$).

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