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Axiomatic Analysis of Unilateral Price Indices

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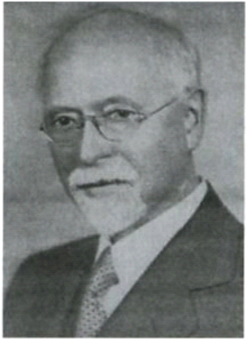
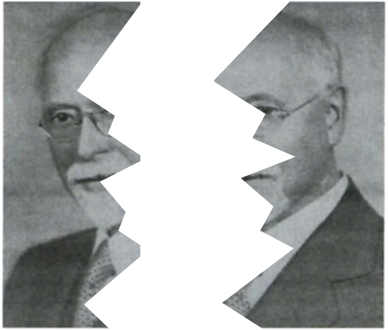
1. Introduction



Irving Fisher (1867-1947)
author of
The Purchasing Power of Money (1911)
The Making of Index Numbers (1922)



macroeconomics:
change of price level

price statistics:
average of price ratios

- Against the notion of a price level (unilateral price index) axiomatic objections have been raised; e.g., Eichhorn and Voeller (1976), Eichhorn (1978), Diewert (1993), and ILO *et al.* (2004).

- Macroeconomic modelling largely ignored this.

- Notation:

$$\begin{array}{rcl} \text{total value of all } N \text{ goods} & : & \sum_{i=1}^{N} p_i x_i \\ \mathbf{p} & = & (p_1, ..., p_N) \\ \mathbf{x} & = & (x_1, ..., x_N)^T \\ \text{unilateral price index} & : & (\mathbf{p}, \mathbf{x}) \longmapsto P(\mathbf{p}, \mathbf{x}) \end{array}$$

2. Tests for Unilateral Price Indices

- Macroeconomists either assume
  - that only one good and therefore only one price exists or
  - that the prices of the economy's heterogeneous products can be aggregated by some *unilateral price index* $P(\mathbf{p}, \mathbf{x})$ into a single number representing the general price level.
- Sensible unilateral price indices must satisfy a list of *tests*.
- Here are some obvious proposals:

T 1 *The **anonymity test** postulates that $P(\mathbf{p}, \mathbf{x})$ is exclusively a function of $\mathbf{p}$ and $\mathbf{x}$.*

T 2 *The **invariance to re-ordering test** postulates, that*

$$P(\mathbf{p}, \mathbf{x}) = P(\tilde{\mathbf{p}}, \tilde{\mathbf{x}}) \ ,$$

where $\tilde{\mathbf{p}}$ and $\tilde{\mathbf{x}}$ are uniform permutations of the vectors $\mathbf{p}$ and $\mathbf{x}$.

T 3 *The **normalizing test** postulates, that for $p_i = p$ $(i = 1, 2, ..., N)$:*

$$P(\mathbf{p}, \mathbf{x}) = p \ .$$

T 4 *The **mean value test** postulates that*

$$\min_i \{p_i\} \leq P(\mathbf{p}, \mathbf{x}) \leq \max_i \{p_i\}$$

In the literature, the following proposals can be found:

T 5 *The* ***positivity test*** *(Diewert, 1993) postulates, that*

$$P(\mathbf{p},\mathbf{x}) > 0 \quad \textit{and} \quad X(\mathbf{p},\mathbf{x}) > 0, \quad \textit{if} \quad \mathbf{p} \gg 0 \quad \textit{and} \quad \mathbf{x} \gg 0 \ .$$

T 6 *The* ***product test*** *(Eichhorn and Voeller, 1976) postulates, that*

$$\sum_{i=1}^{N} p_i x_i = P(\mathbf{p},\mathbf{x}) \cdot X(\mathbf{p},\mathbf{x}) \ .$$

T 7 *The* ***linear homogeneity test*** *(Eichhorn and Voeller, 1976) postulates that*

$$P(\lambda\mathbf{p},\mathbf{x}) = \lambda P(\mathbf{p},\mathbf{x}) \qquad \textit{for all } \lambda > 0 \ .$$

T 8 *The* ***quantity proportionality test*** *(Diewert, 1993) postulates that*

$$P(\mathbf{p}, \lambda\mathbf{x}) = P(\mathbf{p}, \mathbf{x}) \qquad \textit{for all } \lambda > 0 \ .$$

T 9 *The* ***monotonicity test*** *(Eichhorn and Voeller, 1976) postulates that*

$$P(\mathbf{p}, \mathbf{x}) > P(\mathbf{p}^*, \mathbf{x}) \ ,$$

where for all elements $\mathbf{p}$ *and* $\mathbf{p}^*$ *the relation* $p_i^* \geq p_i$ *and for at least one element the strict relation holds.*

T 10 *The* ***strict commensurability test*** *(Eichhorn and Voeller, 1976) postulates that*

$$P(\mathbf{p\Lambda}, \mathbf{x\Lambda^{-1}}) = P(\mathbf{p}, \mathbf{x}) \ ,$$

where $\mathbf{\Lambda}$ *is an arbitrary* $N \times N$ *diagonal matrix with positive elements* λ_i.

3. Proofs of Inconsistency

- No unilateral price index $P(\mathbf{p}, \mathbf{x})$ can exist that simultaneously satisfies tests

  T6 (product)
  T7 (linear homogeneity)
  T9 (monotonicity)
  T10 (strict commensurability) [Eichhorn and Voeller (1976)].

  T6 (product)
  T5 (positivity)
  T7 (linear homogeneity)
  T8 (quantity proportionality)
  T10 (strict commensurability) [Diewert (1993), ILO *et al.* (2004)].

T 11 *The **weak commensurability test** postulates that*

$$P(\mathbf{p}/\lambda, \mathbf{x}\lambda) = P(\mathbf{p}, \mathbf{x}) \qquad \textit{for all } \lambda > 0 \ .$$

- No unilateral price index $P(\mathbf{p}, \mathbf{x})$ can exist that simultaneously satisfies tests

  T7 (linear homogeneity)
  T8 (quantity proportionality)
  T11 (weak commensurability)

  Proof: From T7 and then T8 the following relationship follows:

  $$P(\mathbf{p}/\lambda, \mathbf{x}\lambda) = (1/\lambda)P(\mathbf{p}, \mathbf{x}\lambda) = (1/\lambda)P(\mathbf{p}, \mathbf{x}) \ ,$$

  which contradicts T11

- No unilateral price index $P(\mathbf{p}, \mathbf{x})$ can exist that simultaneously satisfies tests

  T3 (normalizing)
  T11 (weak commensurability)

  Proof: Applied in the sequence T3-T11-T3, these tests generate the contradiction

$$\lambda p = \lambda P(\mathbf{p}, \mathbf{x}) = \lambda P(\mathbf{p}/\lambda, \mathbf{x}\lambda) = \lambda(p/\lambda) = p \; .$$

- Price statisticians have concluded that the search for a suitable unilateral price index should be abandoned.

4. Re-Considering the Proofs

- A sensible unilateral price index P and the corresponding unilateral quantity index X $\left[X = \left(\sum_{i=1}^{N} p_i x_i\right) \Big/ P\right]$ must not satisfy the commensurability test (in contrast to a bilateral price index!).

- Suppose that all N observations relate to the same homogeneous good.

- Decomposition of the total value (product test):

$$\sum p_i x_i = P \cdot X \ . \tag{1.1}$$

- The price component P can be interpreted as an average price level and the quantity component is

$$X = \sum x_i \ . \tag{1.2}$$

- (1.1) and (1.2) give

$$P = \frac{\sum p_i x_i}{X} = \frac{\sum p_i x_i}{\sum x_i} \ .$$

- This is Segnitz's (1870) unit value formula P_{UV}.

- P_{UV} violates the weak and therefore also the strict commensurability test:

$$P_{UV}(\mathbf{p}/\lambda, \mathbf{x}\lambda) = \frac{\sum (p_i/\lambda) x_i \lambda}{\sum x_i \lambda} = \frac{1}{\lambda} \frac{\sum p_i x_i}{\sum x_i} = \frac{1}{\lambda} P_{UV}(\mathbf{p}, \mathbf{x}) \ .$$

- Nevertheless, P_{UV} is the only suitable formula for the price aggregation of a homogeneous good [ILO *et al.* (2004)].

5. Summary

- Price statisticians reject the notion of a general price level (unilateral price index) on the grounds that no formula exists that simultaneously satisfy some indispensable tests together wit the strict commensurability test.

- Here, the case of a homogeneous good was considered.

- For that case it was shown that a unilateral price index must not satisfy the commensurability test.

- However, then the same is true for cases featuring stronger heterogeneity in the goods.

- Macroeconomists were right in ignoring the price statisticians' objections.