Searching for the Holy Grail of Index Number Theory

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Abstract

The index number problem is known as that of decomposing aggregate value change, in ratio or difference form, into two, ideally symmetric, factors. This note comments on a recent contribution of Casler [6].

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1 Introduction

Time and again people are searching for the Holy Grail of index number theory, here defined as being a symmetric pair of price and quantity indices that satisfy all known requirements. Section 2 more precisely describes the

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objective of the search. Section 3 catalogues the findings. Section 4 discusses a recent finding by Casler [6]. The conclusion can be brief: the Holy Grail is a mirage!

2 The index number problem

We consider two time periods, a base period, denoted by the label 0, and a comparison period, denoted by the label 1, and a set of commodities, labeled from 1 to N. The vectors of (unit) prices and quantities of these commodities will be denoted by $p^t \equiv (p_1^t, ..., p_N^t)$ and $q^t \equiv (q_1^t, ..., q_N^t)$ respectively $(t = 0, 1)$. It is assumed that $p^t, q^t \in \Re^N_{++}$. The value of a commodity at period t is then given by $v_n^t \equiv p_n^t q_n^t$ $(n = 1, ..., N; t = 0, 1)$, and the aggregate value by $V^t \equiv \sum_{n=1}^N v_n^t = \sum_{n=1}^N p_n^t q_n^t \equiv p^t \cdot q^t$ $(t = 0, 1)$. The value share of a commodity is defined as $s_n^t \equiv v_n^t / V^t$ $(n = 1, ..., N; t = 0, 1)$. It is clear that

$$\sum_{n=1}^{N} s_n^t = 1 \; (t = 0, 1); \tag{1}$$

that is, the base and comparison period value shares add up to 1.

In the classical index number problem one wants to decompose the aggregate value *ratio* into two parts,

$$\frac{V^1}{V^0} = P(p^1, q^1, p^0, q^0) Q(p^1, q^1, p^0, q^0), \tag{2}$$

of which the first part, $P(p^1, q^1, p^0, q^0)$, measures the effect of differing prices and the second part, $Q(p^1, q^1, p^0, q^0)$, measures the effect of differing quantities. Both functions operate on the price and quantity vectors of the two periods and map these into unitless scalars. Provided that certain reasonable requirements are satisfied, the first part is called a price index and the second part a quantity index.

The indices $P(p^1, q^1, p^0, q^0)$ and $Q(p^1, q^1, p^0, q^0)$ should exhibit the basic properties of continuity, positivity, monotonicity in prices (quantities), linear homogeneity in comparison period prices (quantities), identity, homogeneity of degree zero in prices (quantities), and invariance to the units of measurement (see Balk [1] for precise formulations). The time reversal test stipulates that reversing the time periods yields an index which is identically equal to the reciprocal of the original index. The factor reversal test requires that (2) be satisfied whereby $Q(p^1, q^1, p^0, q^0) = P(q^1, p^1, q^0, p^0)$; that is, price index and quantity index have the same functional form except that prices and quantities have been interchanged. An index is called *ideal* if it satisfies the factor reversal test.

The alternative problem, equally old but lesser known, is to decompose the aggregate value *difference* into two parts,

$$V^1 - V^0 = \mathcal{P}(p^1, q^1, p^0, q^0) + \mathcal{Q}(p^1, q^1, p^0, q^0), \tag{3}$$

of which the first term, $\mathcal{P}(p^1, q^1, p^0, q^0)$, measures the part of the value difference that is due to differing prices and the second term, $\mathcal{Q}(p^1, q^1, p^0, q^0)$, measures the part of the value difference that is due to differing quantities. Both functions operate on the price and quantity vectors of the two periods but map these into money amounts. Provided that certain reasonable requirements are satisfied, the first part is called a price indicator and the second part a quantity indicator.

The indicators $\mathcal{P}(p^1, q^1, p^0, q^0)$ and $\mathcal{Q}(p^1, q^1, p^0, q^0)$ should exhibit the basic properties of continuity, monotonicity in prices (quantities), identity, linear homogeneity in prices (quantities), and invariance to the units of measurement (see Diewert [7] for precise formulations). The time reversal test stipulates that reversing the time periods yields an indicator which is identically equal to the negative of the original indicator. The factor reversal test requires that (3) be satisfied whereby $\mathcal{Q}(p^1, q^1, p^0, q^0) = \mathcal{P}(q^1, p^1, q^0, p^0)$; that is, price indicator and quantity indicator have the same functional form except that prices and quantities have been interchanged. An indicator is called *ideal* if it satisfies the factor reversal test.

The link between additive and multiplicative decompositions is provided by the logarithmic mean.[1] The additive decomposition derived from expression (2) is

$$V^1 - V^0 = L(V^1, V^0) \ln P(p^1, q^1, p^0, q^0) + L(V^1, V^0) \ln Q(p^1, q^1, p^0, q^0). \tag{4}$$

Recall that $L(V^1, V^0)$ is an average of the period 1 value V^1 and the period 0 value V^0, and notice that $\ln P(.)$ and $\ln Q(.)$ are approximately equal to the percentage of aggregate price and quantity change respectively.

[1] The logarithmic mean is, for any two strictly positive real numbers a and b, defined by $L(a, b) \equiv (a - b)/\ln(a/b)$ and $L(a, a) \equiv a$. It has the following properties: (1) $\min(a, b) \leq L(a, b) \leq \max(a, b)$; (2) $L(a, b)$ is continuous; (3) $L(\lambda a, \lambda b) = \lambda L(a, b)$ $(\lambda > 0)$; (4) $L(a, b) = L(b, a)$; (5) $(ab)^{1/2} \leq L(a, b) \leq (a + b)/2$; (6) $L(a, 1)$ is concave.

Reversely, the additive decomposition (3) leads to

$$\frac{V^1}{V^0} = \exp\left\{\frac{\mathcal{P}(p^1, q^1, p^0, q^0)}{L(V^1, V^0)}\right\} \times \exp\left\{\frac{\mathcal{Q}(p^1, q^1, p^0, q^0)}{L(V^1, V^0)}\right\}, \tag{5}$$

as multiplicative decomposition of the value ratio. It is good to notice that properties of indices do not automatically carry over to indicators, and *vice versa*.

3 Ideal indices and indicators

History has provided us with a number of ideal indices. Fisher's [8] solution to the ratio type index number problem was

$$\begin{aligned}\frac{V^1}{V^0} &= \left(\frac{p^1 \cdot q^0}{p^0 \cdot q^0}\frac{p^1 \cdot q^1}{p^0 \cdot q^1}\right)^{1/2}\left(\frac{p^0 \cdot q^1}{p^0 \cdot q^0}\frac{p^1 \cdot q^1}{p^1 \cdot q^0}\right)^{1/2} \\ &\equiv P_F(p^1, q^1, p^0, q^0) Q_F(p^1, q^1, p^0, q^0).\end{aligned} \tag{6}$$

Fisher's indices exhibit all the basic properties, plus the time reversal test, and the factor reversal test.

Montgomery's [9], [10] solution was

$$\begin{aligned}\frac{V^1}{V^0} &= \prod_{n=1}^{N}\left(\frac{p_n^1}{p_n^0}\right)^{L(v_n^1, v_n^0)/L(V^1, V^0)}\prod_{n=1}^{N}\left(\frac{q_n^1}{q_n^0}\right)^{L(v_n^1, v_n^0)/L(V^1, V^0)} \\ &\equiv P_{MV}(p^1, q^1, p^0, q^0) Q_{MV}(p^1, q^1, p^0, q^0).\end{aligned} \tag{7}$$

Since Vartia [14], [15] independently rediscovered this solution to the index number problem, the functions $P_{MV}(.)$ and $Q_{MV}(.)$ are called Montgomery-Vartia indices. These indices satisfy the time reversal test and the factor reversal test. Of the basic properties, they fail to satisfy monotonicity globally. However, as shown by Balk [2], such a failure can hardly be expected to occur in practice. More important is the fact that these indices do not exhibit the basic property of linear homogeneity in comparison period prices (quantities), because of the fact that the weights do not add up to 1 (which in turn depends on the concavity of $L(a, 1)$). Thus this solution is not completely satisfactory.

Sato [12] and Vartia [14], [15] independently discovered a new pair of ideal price and quantity indices. They are given by

$$\begin{aligned}\frac{V^1}{V^0} &= \prod_{n=1}^{N}\left(\frac{p_n^1}{p_n^0}\right)^{L(s_n^1,s_n^0)/\sum_{n=1}^{N}L(s_n^1,s_n^0)}\prod_{n=1}^{N}\left(\frac{q_n^1}{q_n^0}\right)^{L(s_n^1,s_n^0)/\sum_{n=1}^{N}L(s_n^1,s_n^0)} \\ &\equiv P_{SV}(p^1,q^1,p^0,q^0)Q_{SV}(p^1,q^1,p^0,q^0). \qquad (8)\end{aligned}$$

These functions exhibit all the basic index properties except global monotonicity (as shown by Reinsdorf and Dorfman [11]). But, as shown by Balk [2], the failure of monotonicity will only materialize in rather exceptional circumstances. Moreover, the Sato-Vartia indices satisfy the time reversal test as well as the factor reversal test.

The fourth pair of ideal indices was developed by Stuvel [13]. They are not linearly homogeneous in comparison prices (quantities). See Balk [4] for more details on these indices.

We now turn to the difference type index number problem. Bennet's [5] solution was

$$\begin{aligned}V^1 - V^0 &= \frac{1}{2}(q^0+q^1)\cdot(p^1-p^0)+\frac{1}{2}(p^0+p^1)\cdot(q^1-q^0) \\ &= \sum_{n=1}^{N}\frac{q_n^0+q_n^1}{2}(p_n^1-p_n^0)+\sum_{n=1}^{N}\frac{p_n^0+p_n^1}{2}(q_n^1-q_n^0) \\ &\equiv \mathcal{P}_B(p^1,q^1,p^0,q^0)+\mathcal{Q}_B(p^1,q^1,p^0,q^0). \qquad (9)\end{aligned}$$

The Bennet indicators exhibit all the basic properties, plus the time reversal test, and the factor reversal test. The correponding multiplicative decomposition is

$$\frac{V^1}{V^0} = \exp\left\{\frac{\sum_{n=1}^{N}\frac{q_n^0+q_n^1}{2}(p_n^1-p_n^0)}{L(V^1,V^0)}\right\}\times\exp\left\{\frac{\sum_{n=1}^{N}\frac{p_n^0+p_n^1}{2}(q_n^1-q_n^0)}{L(V^1,V^0)}\right\}, \qquad (10)$$

Of the basic index properties these Bennet *indices* fail global monotonicity, as one easily checks, as well as linear homogeneity in comparison period prices (quantities). The two reversal tests remain satisfied.

Montgomery's [9], [10] solution to the difference type index number problem was

$$
\begin{aligned}
V^1 - V^0 &= \sum_{n=1}^{N} \frac{L(v_n^1, v_n^0)}{L(p_n^1, p_n^0)}(p_n^1 - p_n^0) + \sum_{n=1}^{N} \frac{L(v_n^1, v_n^0)}{L(q_n^1, q_n^0)}(q_n^1 - q_n^0) \\
&\equiv \mathcal{P}_M(p^1, q^1, p^0, q^0) + \mathcal{Q}_M(p^1, q^1, p^0, q^0). \qquad (11)
\end{aligned}
$$

The Montgomery indicators satisfy the time reversal test as well as the factor reversal test. Of the basic properties, they only fail to exhibit monotonicity globally, but, as argued by Balk [2], this problem is unlikely to be of practical importance. Applying the transformation given in equation (5) to the Montgomery indicators brings us back to the Montgomery-Vartia indices.

4 Casler's contribution

Against the background sketched in the previous two sections I now turn to Casler's [6] contribution. Basically, Casler neither considers the value ratio V^1/V^0 nor the value difference $V^1 - V^0$, but the asymmetric, forward-looking growth rate $V^1/V^0 - 1$. It is simple to check that the following expression is an identity,

$$
\frac{V^1}{V^0} - 1 = \sum_{n=1}^{N} s_n^0 \frac{\Delta p_n}{p_n^0} + \sum_{n=1}^{N} s_n^0 \frac{\Delta q_n}{q_n^0} + \sum_{n=1}^{N} s_n^0 \frac{\Delta p_n}{p_n^0}\frac{\Delta q_n}{q_n^0}, \qquad (12)
$$

where $\Delta p_n \equiv p_n^1 - p_n^0$ and $\Delta q_n \equiv q_n^1 - q_n^0$ $(n = 1, ..., N)$. Since $\Delta p_n/p_n^0 + \Delta q_n/q_n^0 = \Delta p_n/p_n^0 + \Delta q_n/q_n^0$ $(n = 1, ..., N)$, expression (12) can be replaced by

$$
\frac{V^1}{V^0} - 1 = \sum_{n=1}^{N} s_n^0 \frac{\Delta p_n}{p_n^0} + \sum_{n=1}^{N} s_n^0 \frac{\Delta q_n}{q_n^0} + \sum_{n=1}^{N} s_n^0 \frac{\Delta p_n/p_n^0 + \Delta q_n/q_n^0}{\Delta p_n/p_n^0 + \Delta q_n/q_n^0}\frac{\Delta p_n}{p_n^0}\frac{\Delta q_n}{q_n^0}, \qquad (13)
$$

which can be decomposed as

$$
\begin{aligned}
\frac{V^1}{V^0} - 1 &= \sum_{n=1}^{N} s_n^0 \frac{\Delta p_n}{p_n^0} + \sum_{n=1}^{N} s_n^0 \frac{\Delta p_n/p_n^0}{\Delta p_n/p_n^0 + \Delta q_n/q_n^0}\frac{\Delta p_n}{p_n^0}\frac{\Delta q_n}{q_n^0} \\
&\quad + \sum_{n=1}^{N} s_n^0 \frac{\Delta q_n}{q_n^0} + \sum_{n=1}^{N} s_n^0 \frac{\Delta q_n/q_n^0}{\Delta p_n/p_n^0 + \Delta q_n/q_n^0}\frac{\Delta p_n}{p_n^0}\frac{\Delta q_n}{q_n^0}. \qquad (14)
\end{aligned}
$$

This expression can be simplified to

$$\frac{V^1}{V^0} - 1 = \sum_{n=1}^{N} s_n^0 \left(1 + (1/2)H(\Delta p_n/p_n^0, \Delta q_n/q_n^0)\right) \frac{\Delta p_n}{p_n^0} + \sum_{n=1}^{N} s_n^0 \left(1 + (1/2)H(\Delta p_n/p_n^0, \Delta q_n/q_n^0)\right) \frac{\Delta q_n}{q_n^0}, \qquad (15)$$

where $H(a,b) \equiv 2ab/(a+b)$ denotes the harmonic mean of a and b. This appears to be the decomposition favoured by Casler. The first term at the right-hand side of expression (15) gives the contribution of price change, and the second term gives the contribution of quantity change to $V^1/V^0 - 1$.

Multiplying both sides of this expression by V^0 delivers

$$V^1 - V^0 = \sum_{n=1}^{N} q_n^0 \left(1 + (1/2)H(\Delta p_n/p_n^0, \Delta q_n/q_n^0)\right) \Delta p_n + \sum_{n=1}^{N} p_n^0 \left(1 + (1/2)H(\Delta p_n/p_n^0, \Delta q_n/q_n^0)\right) \Delta q_n. \qquad (16)$$

As Casler observed, these indicators do not satisfy global monotonicity (because of the interaction terms), though in practice that might be a minor problem. Also, the time reversal test is not satisfied.

However, to compare this decomposition to Fisher's, one should proceed one step further and turn expression (16) into a multiplicative decomposition, by using the logarithmic mean transformation. This leads to

$$\frac{V^1}{V^0} = \exp\left\{\frac{\sum_{n=1}^{N} q_n^0 \left(1 + (1/2)H(\Delta p_n/p_n^0, \Delta q_n/q_n^0)\right) \Delta p_n}{L(V^1, V^0)}\right\} \times \exp\left\{\frac{\sum_{n=1}^{N} p_n^0 \left(1 + (1/2)H(\Delta p_n/p_n^0, \Delta q_n/q_n^0)\right) \Delta q_n}{L(V^1, V^0)}\right\}. \qquad (17)$$

These Casler *indices* have several defects. They are not globally monotonic in prices (quantities). They are not linearly homogeneous in comparison period prices (quantities). And they do not satisfy the time reversal test.

It is interesting to observe that the Casler indicators, as in expression (16), are actually members of a family. This family emerges when in expression (13) the ratio $\frac{\Delta p_n/p_n^0 + \Delta q_n/q_n^0}{\Delta p_n/p_n^0 + \Delta q_n/q_n^0}$, which is identically equal to 1, is replaced by $\frac{f(\Delta p_n/p_n^0)+g(\Delta q_n/q_n^0)}{f(\Delta p_n/p_n^0)+g(\Delta q_n/q_n^0)}$ for arbitrary functions $f(a)$ and $g(a)$, which is also identically equal to 1. The Bennet indicators materialize in the case where $f(a) = g(a) = 1$ for all a.

5 Conclusion

It is hard to beat the Fisher indices as decomposition of a value ratio. However, though ideal, they are not perfect. For example, they are not consistent-in-aggregation, and it is not straightforward to represent them as weighted sums or products of individual price (quantity) relatives. On the last point see Balk [3].

The Montgomery-Vartia indices do exhibit consistency-in-aggregation, and are nicely decomposable to the individual commodity level, but they are not linearly homogeneous in comparison prices (quantities).

The Sato-Vartia indices share with Fisher the *in*consistency-in-aggregation, but are nicely decomposable as appears from expression (8). The failure of global monotonicity is practically not very relevant.

Though the Stuvel indices are consistent-in-aggregation, they are decomposition-resistant and not linearly homogeneous in comparison prices (quantities).

Though performing perfectly as indicators, the Bennet *indices* fail global monotonicity as well as linear homogeneity in comparison prices (quantities).

Both the Casler indicators and indices fail global monotonicity, and time reversal. In addition the Casler indices fail linear homogeneity in comparison price (quantities).

That the Holy Grail of index number theory has as yet not been found, does not come as a surprise. The thing does not exist, as substantiated extensively by Balk [4]. But the quest continues to deliver interesting findings.

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