Questioning Some General Wisdom in Axiomatic Index Theory

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Abstract: Hitherto, axiomatic index theory draws a strict and artificial dividing line between price aggregations in the context of heterogeneous items and price aggregations in the context of homogeneous items. However, each price aggregation problem should be viewed as a specific case in a continuum of cases that range from complete homogeneity to strong heterogeneity. This continuity approach generates some surprising new insights that reverse important parts of what has been regarded as the general wisdom of axiomatic index theory.

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1 Introduction

Since the seminal contributions of Walsh (1901) and Fisher (1911, 1922), axiomatic index theory has been one of the major fields of research in price statistics. It is primarily concerned with the development of formal postulates that any sensible price index formula must satisfy. These formal postulates are usually denoted as tests or axioms. Recent surveys of axiomatic index theory include Balk (1995), ILO *et al.* (2004), and Reinsdorf (2007).

Hitherto, axiomatic index theory draws a strict and artificial dividing line between price aggregations in the context of heterogeneous items and price aggregations in the context of homogeneous items. However, each price aggregation problem should be viewed as a specific case in a continuum of cases that range from complete homogeneity to strong heterogeneity. Following this continuity approach, one gains some surprising new insights that reverse important parts of what one could regard as the general wisdom of axiomatic index theory.

The basic idea of the continuity approach is explained in Section 2. Building upon this idea, in Section 3 it is argued that the so called identity axiom is flawed. Reasons for the identity axiom's ongoing popularity are discussed in Section 4. Some related axioms that have been regarded as indispensable, are examined in Section 5. In Section 6 new axioms are proposed that, in the light of the new approach, represent sensible postulates for meaningful price indices. They also provide an axiomatic justification for the use of the unit value index in situations with homogeneous items. Section 7 provides some concluding remarks.

2 The Price Measurement Problem: Three Levels of Complexity

This study is concerned with measuring the overall price change between two periods. They are denoted as base period and comparison period. The primal entity to be considered should be the individual transaction between seller and buyer. Dalén (2001, p. 3) points out that, by definition, a transaction taking place during the base period can never have an exactly matching transaction in the comparison period. Therefore, intertemporal price comparisons require to group the set of transactions into N subsets of transactions. The transactions belonging to some subset i ($i = 1, 2, .., N$) must be sufficiently similar. Similarity refers not only to the type of good (or service) but also to other aspects such as the location and the point of time (within the period) at which the good was purchased. Adding up the quantities associated with the transactions of period t belonging to subset i yields the quantity x_i^t. Similarly, v_i^t represents the total expenditures associated with the transactions of period t belonging to subset i. Finally, dividing total expenditures v_i^t by the quantity x_i^t gives the unit value p_i^t of subset i during period t.

In axiomatic index theory it is assumed that for each of the N subsets in both the base and the comparison period at least one transaction exists. As a consequence, the unit values and quantities of period t can be listed in the row vector $\mathbf{p^t} = (p_1^t, ..., p_N^t)$ and in the column vector $\mathbf{x^t} = (x_1^t, ..., x_N^t)^T$ with $t = 0$ (base period) and $t = 1$ (comparison period). Formally, a price index P is a function that maps the vectors $\mathbf{p^0}$, $\mathbf{x^0}$, $\mathbf{p^1}$, and $\mathbf{x^1}$ in a positive real number $P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1})$:

$$P : \mathbb{R}^{4N}_{++} \longmapsto \mathbb{R}_{++} , \quad (\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \longmapsto P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) . \tag{1}$$

The unit values and quantities are regarded as independent from each other. For simplicity, in the rest of the paper p_i^t is called the price (instead of unit value) of item i (instead of subset i) in period t.

If in the price measurement problem only one item existed, then the overall price change could be directly computed from the price ratio (p_1^1/p_1^0). Unfortunately, the complexity of real world price measurement problems is usually much higher. The level of complexity can be viewed as a function of two dimensions:

1. the number of items and
2. the degree of heterogeneity of the items considered.

Every price measurement problem can be characterized as a combination of these two features. In traditional axiomatic index theory the set of all possible combinations are subdivided into three distinct subsets. They are illustrated in Figure 1. Since each degree of heterogeneity requires some minimum number of items, the points on the vertical axis and on the neighbouring white area do not belong to any of the three subsets.

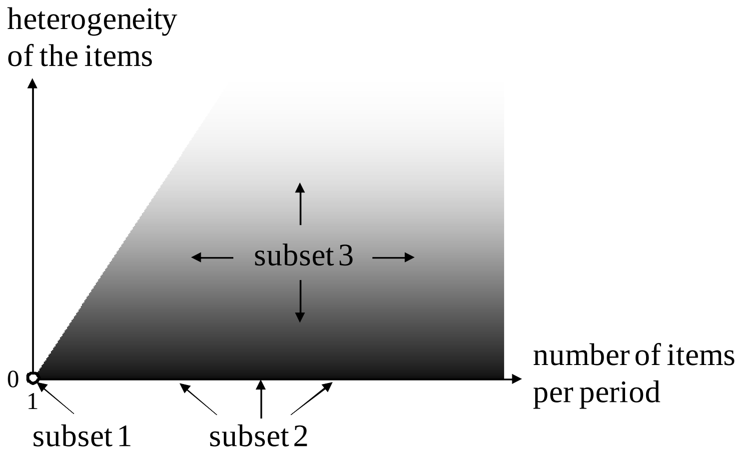

Figure 1: Subdividing the Set of Price Measurement Problems into Three Subsets.

Subset 1: Single Item $(\mathbf{N = 1})$

This is the simplest possible price measurement problem. In Figure 1 this subset is represented by the intersection of the horizontal and the vertical axis. The overall price change P is given by the price ratio of the (single) item:

$$P = \frac{p_1^1}{p_1^0} . \tag{2}$$

Subset 2: Homogeneous Items $(\mathbf{N > 1})$

Such a measurement problem arises when the transactions not only within each of the N items but also across these items are sufficiently similar. In Figure 1, this group of price measurement problems is represented by the points on the horizontal axis (without the intersection with the vertical axis), where, for the sake of simplicity, the number of items is interpreted as a continuous variable. In ILO *et al.* (2004, p. 164) and Balk (2005, p. 678) it is recommended that in this subset of price measurement problems Drobisch's (1871, p. 39) *unit value index* should be used:

$$P_{UV} = \frac{\left(\sum p_i^1 x_i^1\right) / \left(\sum x_i^1\right)}{\left(\sum p_i^0 x_i^0\right) / \left(\sum x_i^0\right)} \ , \tag{3}$$

where $\sum = \sum_{i=1}^{N}$.

Subset 3: Heterogeneous Items $(\mathbf{N > 1})$

With heterogeneous items the number of observed items is necessarily larger than one. Graphically, this subset is represented by the grey shaded area (without the horizontal axis). The subset comprises a wide variety of cases. The fading grey emphasizes that the cases of this subset within splitting distance from the horizontal axis have much more in common with the cases of subset 2 than with those cases of its own subset that are characterized by strongly heterogeneous items. Nevertheless, traditional axiomatic index theory has treated all cases of subset 3 as being alike and completely unrelated to subset 2. In spite of this common approach, there is an ongoing controversy as to which price index formula P should be applied to the cases of subset 3. In ILO *et al.* (2004, p. 325, p. 357) the Fisher, Walsh, or Törnqvist index are recommended for all cases belonging to subset 3. All price indices mentioned in this study and some additional price indices are listed in the Appendix.

3 Identity Axiom

A price index $P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1})$ aggregates the available price and quantity information into some overall price change. A meaningful price index must conform to several formal postulates. These postulates are usually denoted as axioms or tests. There is much controversy as to which axioms a price index must satisfy. Some axioms have been repeatedly criticized, others have not aroused severe objections. To this latter group belongs the identity axiom proposed by Laspeyeres (1871, p. 308).

Identity Axiom:

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^0}, \mathbf{x^1}) = 1 \ .$$

This axiom is regarded as an obvious and compelling requirement that any sensible price index must satisfy (e.g., Walsh 1901, p. 115; Eichhorn and Voeller, 1976, p. 24). It postulates that with constant prices the price index should show "no overall price change", regardless of any changes in the quantities. Table 1 provides

Table 1: Prices and Quantities of Four Items.

	base period		comparison period	
	price	quantity	price	quantity
item A	8	57	8	56
item B	9	50	9	2
item C	8	43	8	40
item D	7	50	7	102

an example. None of the prices has changed between the base and the comparison period. The quantities, however, have shifted towards the cheapest item.

When items A to D represent the same homogeneous good, the price measurement problem belongs to subset 2. Following Balk (2005, p. 678) and the official recommendations of ILO *et al.* (2004, p. 164), the overall price change of these four items should be computed on the basis of the unit value index (3). The resulting index number is $P_{UV} = 7.5\,/\,8 = 0.9375$, indicating a fall in overall prices by 6.25 per cent – in contrast to what the identity axiom postulates. An axiomatic justification for the use of the unit value index (3) is provided in Section 6.

The example illustrates that in the context of subset 2 (homogeneous items) the identity axiom's postulate is inadequate. The same critical position can be found in Silver (2008, p. 6). In defence of the identity axiom he points out that this axiom and all other axioms have never been concerned with subset 2, but exclusively apply to price measurement problems of subset 3 (heterogeneous items). Therefore, it would be sufficient that these axioms are meaningful in the context of heterogeneous items. Unfortunately, the identity axiom remains inadequate also in the context of subset 3. In order to see this, one can borrow from mathematics a simple continuity argument.

As pointed out before, the price measurement problems depicted in Figure 1 represent a continuum of cases. The points on the horizontal axis represent cases of complete homogeneity (subset 2). When the prices of the homogeneous items remain constant but their quantities change, then this implies expenditure shifts. If these shifts occur towards the items with lower prices, say, then this should result in a price index number that indicates a decline in the general price level of the homogeneous items. The unit value index satisfies this postulate. The universal recommendation of the unit value index documents that for the price measurement of homogeneous items agreement exists that quantities should matter even if all prices remained constant. In other words, the applied price index should violate the identity axiom.

Now consider exactly the same scenario, the only difference being that the observed items are no longer perfectly but merely almost perfectly homogeneous. In Figure 1, these cases are represented by the points within splitting distance from the horizontal axis. If quantity and therefore expenditure shifts mattered for the price measurement of completely homogeneous items with constant prices, why should they suddenly become less relevant when the items are no longer completely homogeneous but merely almost perfectly homogeneous? In fact, if one followed the identity axiom, quantities would suddenly become *completely* irrelevant. However, there is no reason why the trace of homogeneity should have any such effect. If expenditure shifts matter for perfectly homogeneous items with constant prices, then they also matter for almost homogeneous items with constant prices. In sum, the identity axiom remains inadequate whether the items are completely homogeneous or not.[2]

This argument could be taken even further. There is also no reason why with increasing degree of heterogeneity expenditure shifts should become less relevant. Even in the case of strongly heterogeneous goods with constant prices expenditure shifts matter, and thus, the identity axiom remains inadequate. Those upholding the view that with sufficiently heterogeneous goods the deficiency of the identity axiom vanishes should come up with a convincing rule for the degree of heterogeneity sufficient for the quantities no longer to matter.

4 Reasons for the Identity Axiom's Popularity

What are the reasons for the identity axiom's ongoing popularity? In price statistics, it is widely believed that only those price index formulas are sensible price index formulas that represent some form of averaging of the individual items' intertemporal price ratios. The identity axiom is concerned with a situation in which all these price ratios have the value one. Therefore, also the average of the price ratios necessarily is one. In other words, the identity axiom is a natural extension of the principle of averaging the items' individual price ratios.

If one accepts the view that the identity axiom is flawed, an interesting reversal of the previous argument arises: Since the identity axiom is flawed, all price index formulas satisfying this axiom should be viewed with suspicion. All formulas that average the individual price ratios necessarily satisfy the identity axiom. Therefore, all these formulas should be viewed with suspicion. In fact, in Auer (2008) a new class of price index formulas is developed that do not average the individual items' price ratios. This class is denoted as the family of generalized unit value indices. Some of its members have axiomatic profiles that stand up to those of the most highly regarded "traditional" price indices (e.g., Fisher and Walsh index).

A second reason for the popularity of the identity axiom is the strict though artificial dividing line that axiomatic index theory draws between considerations in the context of subset 2 (homogeneous items) and subset 3 (heterogeneous items). Insights gained in the context of subset 2 are regarded as completely irrelevant for subset 3. Furthermore, considerations derived in the context of subset 3 are viewed as being equally applicable to all cases within this subset and as completely irrelevant in the context of subset 2. As pointed out before, however, subset 3 comprises a continuum of cases ranging from almost homogeneous items to extremely heterogeneous items. The cases involving almost homogeneous items have far more in common with subset 2 than with those cases of subset 3 characterized by strongly heterogeneous items. Insights gained from subset 2 are therefore relevant for subset

[2] Silver (2008) is concerned with the case of almost homogeneous items and the appropriate price index formula. Deviating from the usual recommendations (Fisher, Törnqvist, or Walsh index), he suggests to aggregate the available price and quantity information by a weighted average of the Fisher index and the unit value index. Such a price index violates the identity axiom and this is a strength rather than a weakness of the price index proposed by Silver.

3. Conversely, conclusions drawn from subset 3 should remain valid also in the context of subset 2.

This implies that axioms must make sense in the context of all three subsets, not just in one or two of them. Some of the standard axioms (e.g., monotonicity axiom) satisfy this requirement supporting their relevance and importance in the evaluation of price index formulas. The identity axiom, however, does not belong to this group of axioms. In the context of subset 2 it was clearly flawed. Invoking continuity considerations has revealed that this axiom remains flawed also in the context of subset 3.

Axiomatic index theory has yet not sufficiently exploited the potential of insights that, though gained in the context of subset 2, can be applied to subset 3. Only with this in mind, one can understand why, without notable discussion, the identity axiom has kept its place in the list of indispensable axioms that a price index should satisfy. If one accepts that the identity axiom is flawed, then this extends also to any tightening of this axiom. When a price index formula satisfies not only the identity axiom but even tightenings of this axiom, then this is an additional weakness rather than a particular strength of this price index formula. The following section is concerned with such tightenings.

5 Tightenings of the Identity Axiom

The previous considerations have demonstrated that shifts in the quantity structure should affect the value of the price index – regardless of whether all prices have remained constant and regardless of whether the prices of homogeneous or heterogeneous items are to be aggregated. In the past, axioms have been formulated that seemingly aim at this feature. One of these axioms was proposed by Krtscha (1979, p. 66).

Gravitation Axiom: The mapping

$$d_i(x_i^0, x_i^1) := \left| P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) - \frac{p_i^1}{p_i^0} \right| \tag{4}$$

is weakly monotonically falling in the variables x_i^0 and x_i^1.

The axiom postulates that an increase in the quantities x_i^0 or x_i^1 must not bring the value of the price index further away from the value of the price ratio of item i. Whereas the Laspeyres and Paasche index satisfy this axiom, the Fisher index violates it.

To see more clearly the relationship between the gravitation axiom and the identity axiom, the following simple test is introduced:

Normalizing Axiom:

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^0}, \mathbf{x^0}) = 1 \; . \tag{5}$$

If one restricts the gravitation axiom to the scenario $\mathbf{p^0} = \mathbf{p^1}$ (the identity axiom's scenario), Equation (4) simplifies to

$$d_i(x_i^0, x_i^1) := \left| P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^0}, \mathbf{x^1}) - 1 \right| \; . \tag{6}$$

Price indices that satisfy the normalizing axiom also satisfy the relationship

$$\left|P(\mathbf{p^0},\mathbf{x^0},\mathbf{p^0},\mathbf{x^0}) - 1\right| = 0 \ . \tag{7}$$

For the case $\mathbf{x^0} = \mathbf{x^1}$, Equations (7) and (6) give

$$d_i(x_i^0, x_i^1) = 0 \ . \tag{8}$$

The gravitation axiom postulates that the function $d_i(x_i^0, x_i^1)$ keeps the value 0 when the quantities x_i^0 and x_i^1 are changed. This includes quantity changes leading to $\mathbf{x^0} \neq \mathbf{x^1}$. In other words, for price index formulas that satisfy the normalizing axiom, the gravitation axiom postulates that $P(\mathbf{p^0},\mathbf{x^0},\mathbf{p^0},\mathbf{x^0}) = 1$ and also that $P(\mathbf{p^0},\mathbf{x^0},\mathbf{p^0},\mathbf{x^1}) = 1$. The latter is precisely the postulate of the identity axiom. Therefore, for price indices that satisfy the normalizing axiom (e.g., all formulas listed in the Appendix), the gravitation axiom is a tightening of the identity axiom and not, as originally conjectured, a test inconsistent with the identity axiom.

An obvious tightening of the identity axiom has been proposed by Walsh (1901, p. 115).

Proportionality Axiom:

$$P(\mathbf{p^0},\mathbf{x^0},\lambda\mathbf{p^0},\mathbf{x^1}) = \lambda \ , \quad \text{for all } \lambda > 0 \ .$$

The axiom considers scenarios in which over the two periods all prices change in the same proportion: $\mathbf{p^1} = \lambda\mathbf{p^0}$. In such scenarios the value of the price index should be equal to the factor of proportionality λ. Changes in quantities should be irrelevant. Restricting the proportionality axiom to the case $\lambda = 1$, one obtains the postulate of the identity axiom. Whereas all traditional price indices satisfy the identity axiom, two of them (Walsh-Vartia and Vartia index) violate the proportionality axiom.

A stronger tightening of the identity axiom is the following proposal by Eichhorn and Voeller (1976, p. 28).

Mean Value Axiom:

$$\min_i \frac{p_i^1}{p_i^0} \leq P(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\mathbf{x^1}) \leq \max_i \frac{p_i^1}{p_i^0} \ .$$

According to this axiom the value of the price index should always lie between the smallest and largest intertemporal price ratio. Restricting the mean value axiom to the case in which all prices change in the same proportion ($\mathbf{p^1} = \lambda\mathbf{p^0}$), one obtains the postulate of the proportionality axiom.

An alternative tightening of the identity axiom amalgamates proposals by Eichhorn and Voeller (1976, p. 24 and p. 28).

Linear Homogeneity Axiom:

$$P(\mathbf{p^0},\mathbf{x^0},\lambda\mathbf{p^1},\mathbf{x^1}) = \lambda P(\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\mathbf{x^1}) = P((1/\lambda)\mathbf{p^0},\mathbf{x^0},\mathbf{p^1},\mathbf{x^1}), \ \text{for all } \lambda > 0 \ .$$

This axiom postulates that a proprotional change of the comparison period prices or the base period prices should change the value of the price index by the same proportion. Restricting this postulate to the case $\mathbf{p^1} = \mathbf{p^0}$, one obtains the postulate of the proportionality axiom. As a consequence, the Walsh-Vartia and the Vartia index violate not only the proportionality axiom and the mean value axiom but also the linear homogeneity axiom.

In Section 4 it was outlined why the identity axiom is flawed. For those agreeing with this position, the previously listed tightenings of the identity axiom (proportionality axiom, mean value axiom, linear homogeneity axiom) are even less compelling. Those accepting the normalizing axiom should also reject the gravitation axiom, because, given the validity of the normalizing axiom, the gravitation axiom is another tightening of the identity axiom.

6 Further Axiomatic Considerations

In Section 2 it was pointed out that between the cases of subset 2 (homogeneous items) and those cases of subset 3 characterized by extremely heterogeneous items, a continuum of cases exists all belonging to subset 3. Some of these cases are close to subset 2. Invoking continuity considerations, it was argued that postulates formulated in the context of subset 3 are only compelling, if they also make sense in the context of subset 2. Conversely, if a postulate is inadequate in the context of subset 2, it is also inadequate in the context of subset 3. Using this line of reasoning, it was concluded that the satisfaction of the identity axiom is a deficiency rather than a virtue of a price index.

The close relationship between subsets 2 and 3 can be utilized in an even more general way. Furthermore, one can also exploit the relationship between subset 1 (a single item) and subsets 2 and 3. This approach leads to new compelling axioms that are the object of this section. The analysis also develops an axiomatic justification for the recommendation that in subset 2 the unit value index (3) should be used.

For a price index to be meaningful it is necessary that for the case of a single item (subset 1) the index formula simplifies to the price ratio (2). All of the traditional price indices satisfy this requirement. It is also satisfied by the unit value index (3). Following Eichhorn and Voeller (1990, p. 326), the requirement can also be formulated as a formal axiom.

Price Ratio Axiom: For $N = 1$, the following relationship must be satisfied:

$$P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) = \frac{p_1^1}{p_1^0} \, .$$

To which index formula should a general price index simplify when applied to the special case of homogeneous items, that is, to subset 2? In contrast to the aggregation of heterogeneous items, for the aggregation of homogeneous items a natural measure of the quantities exist. Related to a single period t, this measure is

$$X^t := \sum x_i^t \, . \tag{9}$$

In such a case it makes sense to decompose total expenditure $\sum p_i^t x_i^t$ into the quantity component defined in (9) and some price component P^t:

$$\sum p_i^t x_i^t = P^t \sum x_i^t \ .$$

Solving for P^t yields the price component

$$P_{UV}^t = \frac{\sum p_i^t x_i^t}{\sum x_i^t} \ . \tag{10}$$

The subscript UV has been added to emphasize that this price component is the unit value introduced by Segnitz (1870, p. 184),

Having computed the unit value P_{UV}^t, the N homogeneous items can be further processed as a single item with the uniform price P_{UV}^t and the quantity $X^t(= \sum x_i^t)$. Utilizing Equations (9) and (10), a case of subset 2 has been transformed into the case defined as subset 1. This is useful when deriving the suitable price index formula for measuring the overall price change for homogeneous items (subset 2). Since one has transformed a subset 2 price measurement problem into a subset 1 price measurement problem, the price ratio axiom can be invoked. As a consequence, the overall price change should be computed from the price ratio

$$P = \frac{P_{UV}^1}{P_{UV}^0} = \frac{\left(\sum p_i^1 x_i^1\right) / \left(\sum x_i^1\right)}{\left(\sum p_i^0 x_i^0\right) / \left(\sum x_i^0\right)} \ .$$

This is the unit value index P_{UV}, defined in Equation (3).

In summary, the following statement can be made: If one agrees with the postulate of the price ratio axiom and simultaneously regards the unit value (10) as the appropriate measure for calculating the average price of homogeneous items, then the calculation of the overall price change of homogeneous items (subset 2) must be based on the unit value index (3).

In the context of heterogeneous items, the unit value index (3) exhibits serious shortcomings and therefore should be avoided. For example, it violates the following axiom proposed by Pierson (1896, p. 131).

Strict Commensurability Axiom:

$$P(\mathbf{p^0 \Lambda}, \mathbf{x^0 \Lambda^{-1}}, \mathbf{p^1 \Lambda}, \mathbf{x^1 \Lambda^{-1}}) = P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \ ,$$

where $\mathbf{\Lambda}$ is an arbitrary $N \times N$ diagonal matrix with positive elements λ_i.

This axiom postulates that the chosen units of measurement should not affect the value of the price index. The elements λ_i define by which factor the units of measurement have been changed. For example, a change from kilogramm to gramm implies that $\lambda_i = 1/1000$. This test is satisfied by all traditional price indices.

In the context of homogeneous items, however, the strict commensurability test is not relevant. Changing the units of measurement of homogeneous items implies that all elements λ_i have the same value. Relevant in such a context is not the strict commensurability axiom but a weaker version of this axiom that has been proposed by Swamy (1965, p. 620).

Weak Commensurability Axiom:

$$P(\mathbf{p^0}\lambda, \mathbf{x^0}\lambda^{-1}, \mathbf{p^1}\lambda, \mathbf{x^1}\lambda^{-1}) = P(\mathbf{p^0}, \mathbf{x^0}, \mathbf{p^1}, \mathbf{x^1}) \ ,$$

where $\lambda > 0$.

This axiom is satisfied by all traditional price indices but also by the unit value index.

Also the following axiom justifies the unit value index as the appropriate price index for subset 2.

Permutation Axiom: The vectors $\widetilde{\mathbf{p}}^{\mathbf{0}}$ and $\widetilde{\mathbf{x}}^{\mathbf{0}}$ are uniform permutations of the vectors $\mathbf{p^0}$ and $\mathbf{x^0}$. Then, the following relationship must be satisfied:

$$P(\mathbf{p^0}, \mathbf{x^0}, \widetilde{\mathbf{p}}^{\mathbf{0}}, \widetilde{\mathbf{x}}^{\mathbf{0}}) = 1 \ .$$

This test has been advocated in Auer (2002). The test is satisfied by the unit value index (3), but not by the traditional price indices listed in the appendix.

This section has axiomatically confirmed that in the context of subset 2 the overall price change should be computed on the basis of the unit value index. This recommendation is in line with the official recommendation given in ILO *et al.* (2004, p. 164).

7 Concluding Remarks

In traditional axiomatic index theory there exists a strict dividing line between price measurement in the context of heterogeneous items and price measurement in the context of homogeneous items. As a consequence, insights from one context have been regarded as completely irrelevant for the other context. However, this strict dividing line is completely arbitrary and artificial. Between the context of homogeneous items and the context of strongly heterogeneous items a continuum of cases exists. Cases with weakly heterogeneous items have much more in common with the case of full homogeneity than with the case of strong heterogeneity. Removing the artificial dividing line, one obtains important new insights that confirm some and reverse other aspects of what has been regarded as the general wisdom of axiomatic index theory.

Confirming the recommendation given in ILO *et al.* (2004, p. 164), this study has formally demonstrated that the unit value index is the best index formula for the price aggregation in the context of homogeneous items. Furthermore, this study argued that a meaningful aggregation of homogeneous items requires that the result depends not only on the items' prices but also on their quantities, even when the prices do not change over time. The unit value index satisfies this requirement. As a consequence, this index violates the so called identity axiom. The violation of the identity axiom is a strength rather than a weakness of the unit value index.

When quantities matter in the context of homogeneous items with constant prices, then they also must matter for weakly heterogeneous items with constant prices and therefore also for strongly heterogeneous items with constant prices. As a consequence of this simple continuity argument, the identity axiom appears flawed also in the context of price aggregation of heterogeneous items. Nevertheless all traditional price indices satisfy the identity test. This reveals a common weakness in all of these tests and suggests to look for alternative price indices. Such indices are developed in Auer (2008).

Appendix

Expenditures on item i in period t are $v_i^t = p_i^t x_i^t$ and total expenditures on all items in period t are $V_i^t = \sum v_i^t$, where $\sum = \sum_{i=1}^{N}$. Correspondingly, the quantities of period t evaluated with the prices of period s are $v_i^{st} = p_i^s x_i^t$ and $V_i^{st} = \sum v_i^{st}$.

The two most popular price index formulas are:

$$\begin{aligned} \text{Laspeyres}: \ P_L &= \frac{\sum p_i^1 x_i^0}{\sum p_i^0 x_i^0} = \frac{V^{10}}{V^0} \\ \text{Paasche}: \ P_P &= \frac{\sum p_i^1 x_i^1}{\sum p_i^0 x_i^1} = \frac{V^1}{V^{01}} \ , \end{aligned}$$

Several alternativ price index formulas exist:

$$\begin{aligned} \text{Fisher}: \ P_F &= \sqrt{P_L P_P} = \sqrt{\frac{V^{10} V^1}{V^0 V^{01}}} \\ \text{Drobisch}: \ P_{Dr} &= \frac{1}{2}\left(P_L + P_P\right) \\ \text{Marshall–Edgeworth}: \ P_{ME} &= \frac{\sum p_i^1 (x_i^0 + x_i^1)}{\sum p_i^0 (x_i^0 + x_i^1)} = \frac{V^{10} + V^1}{V^0 + V^{01}} \\ \text{Walsh(I)}: \ P_{W(I)} &= \frac{\sum p_i^1 \sqrt{x_i^0 x_i^1}}{\sum p_i^0 \sqrt{x_i^0 x_i^1}} = \frac{\sum \sqrt{v_i^{10} v_i^1}}{\sum \sqrt{v_i^0 v_i^{01}}} \ . \end{aligned}$$

Furthermore, some log change price index formulas exist:

$$\begin{aligned} \text{Walsh(II)}: \ \ln P_{W(II)} &= \sum \frac{\sqrt{v_i^0 v_i^1}}{\sum \sqrt{v_j^0 v_j^1}} \ln \frac{p_i^1}{p_i^0} \\ \text{Törnqvist}: \ \ln P_T &= \sum \frac{(v_i^0/V^0) + (v_i^1/V^1)}{2} \ln \frac{p_i^1}{p_i^0} \\ \text{Theil}: \ \ln P_{Th} &= \sum \left[\frac{\sqrt[3]{\frac{1}{2}\left(v_i^0 + v_i^1\right) v_i^0 v_i^1}}{\sum \sqrt[3]{\frac{1}{2}\left(v_j^0 + v_j^1\right) v_j^0 v_j^1}} \right] \ln \frac{p_i^1}{p_i^0} \\ \text{Walsh–Vartia}: \ \ln P_{WV} &= \sum \frac{\sqrt{v_i^0 v_i^1}}{\sqrt{V^0 V^1}} \ln \frac{p_i^1}{p_i^0} \\ \text{Vartia}: \ \ln P_{V(I)} &= \frac{\ln V^1 - \ln V^0}{V^1 - V^0} \sum \frac{v_i^1 - v_i^0}{\ln v_i^1 - \ln v_i^0} \ln \frac{p_i^1}{p_i^0} \ , \\ &\text{where } \left(v_i^1 - v_i^0\right) / \left(\ln v_i^1 - \ln v_i^0\right) =_i^0 \text{ if } v_i^1 = v_i^0 \\ &\text{and } \left(\ln V^1 - \ln V^0\right) / \left(V^1 - V^0\right) = 1/V^0 \text{ if } V^1 = V^0 \ . \end{aligned}$$

Two additional price index formulas are

$$\text{Banerjee} \quad : \quad P_B = P_P \frac{P_L + 1}{P_P + 1} = \frac{V^{10}/V^0 + 1}{1 + V^{01}/V^1}$$

$$\text{Stuvel} \quad : \quad P_S = \frac{Z}{2} + \sqrt{\left(\frac{Z}{2}\right)^2 + \frac{V^1}{V^0}}$$

$$\text{with } Z = \left(V^{10} - V^{01}\right)/V^0 \ .$$

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