Axiomatic Analysis of Unilateral Price Indices

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Abstract: Unilateral price indices are price index formulas that aggregate the observed prices and quantities of a period into some sort of average price level. Their use has been discouraged on the grounds that they necessarily violate certain formal requirements that have been regarded as indispensable for a meaningful unilateral price index. The present study challenges this position.

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1 Introduction

Unilateral price indices are price index formulas that aggregate the observed prices and quantities of a period into some sort of average price level. Since the classic works of Walsh (1901) and Fisher (1922), the use of such unilateral price indices has been discouraged. The first formal justification for this position has been contributed by Eichhorn and Voeller (1976, pp. 75-78). They define a set of basic logical requirements that any sensible unilateral price index formula must satisfy. In a second step, they show that no unilateral price index formula can exist that satisfies all these requirements. Similar impossibility theorems can be found in Eichhorn (1978, pp. 144-46), Diewert (1993, pp. 7-9), and ILO *et al.* (2004, p. 292). The present study explains why all these findings are not convincing.

Section 2 introduces the notion of a unilateral price index and Section 3 lists a number of basic requirements that a unilateral price index should satisfy. Section 4 sketches out the impossibility theorems put forward against the notion of a unilateral price index and it adds two even stronger impossibility theorems. The common defect of all these impossibility theorems is explained in Section 5. Section 6 concludes.

2 Unilateral Price Indices

In the following it is assumed that N different items can be purchased during period t. The price of item i in period t is denoted by p_i^t. Correspondingly, x_i^t is the quantity and $p_i^t x_i^t$ is the (monetary) value of transactions in item i during period t. The quantities of the N items are represented by the column vector $\mathbf{x}^t = (x_1^t, ..., x_N^t)^T$ and the corresponding prices by the row vector $\mathbf{p}^t = (p_1^t, ..., p_N^t)$. As is common in price index theory, prices and quantities are considered as being independent from each other. A unilateral price index P measures the general price level of the period considered. Formally, P is a function that maps the two vectors $\mathbf{p}^t$ and $\mathbf{x}^t$ into a positive real index number $P(\mathbf{p}^t, \mathbf{x}^t)$:

$$P : \mathbb{R}_{++}^{2N} \longmapsto \mathbb{R}_{++} \ , \qquad (\mathbf{p}^t, \mathbf{x}^t) \longmapsto P(\mathbf{p}^t, \mathbf{x}^t) \ .$$

Correspondingly, a unilateral quantity index X is defined as

$$X : \mathbb{R}_{++}^{2N} \longmapsto \mathbb{R}_{++} \ , \qquad (\mathbf{p}^t, \mathbf{x}^t) \longmapsto X(\mathbf{p}^t, \mathbf{x}^t) \ .$$

Sometimes it is desirable to provide users with a time series of unilateral price indices ($t = 1, 2, .., \tau$). Of course, this time series could be multiplied by some constant c such that some period s is used as the reference period. For example, the time series

$$\tilde{P}(\mathbf{p}^t, \mathbf{x}^t) = c\, P(\mathbf{p}^t, \mathbf{x}^t)$$

with

$$c = \frac{100}{P(\mathbf{p}^s, \mathbf{x}^s)} \qquad \text{and} \qquad t = 1, 2, ..., \tau \ ,$$

would assign the value 100 to the price level of the reference period s. The difference $\tilde{P}(\mathbf{p}^t, \mathbf{x}^t) - 100$ could be interpreted as the percentage change between period t and the reference period s. The "derived" index $\tilde{P}(\mathbf{p}^t, \mathbf{x}^t)$ is merely a proportional transformation of the "genuine" unilateral price index $P(\mathbf{p}^t, \mathbf{x}^t)$.

3 Axioms for "Genuine" Unilateral Price Indices

As pointed out by Eichhorn and Voeller (1976, p. 76), sensible "genuine" unilateral price indices must satisfy a list of basic formal requirements that are denoted here as axioms. Proposals for such axioms are presented in the following list. The meaning of each axiom is explained right afterwards. The superscripts t are dropped

A 1 *The* ***anonymity axiom*** *postulates that* $P(\mathbf{p},\mathbf{x})$ *is exclusively a function of* $\mathbf{p}$ *and* $\mathbf{x}$*.*

A 2 *The* ***invariance to re-ordering axiom*** *postulates that*

$$P(\mathbf{p},\mathbf{x}) = P(\tilde{\mathbf{p}},\tilde{\mathbf{x}}) \ ,$$

where $\tilde{\mathbf{p}}$ *and* $\tilde{\mathbf{x}}$ *are uniform permutations of the vectors* $\mathbf{p}$ *and* $\mathbf{x}$*.*

A 3 *The* ***single observation axiom*** *postulates that for* $N = 1$*:*

$$P(\mathbf{p},\mathbf{x}) = p_1 \ ,$$

where p_1 *is the price of the observed item.*

A 4 *The* ***uniformity axiom*** *postulates that for* $p_i = p$ $(i = 1, 2, ..., N)$*:*

$$P(\mathbf{p},\mathbf{x}) = p \ .$$

A 5 *The* ***mean value axiom*** *postulates that*

$$\min_i\{p_i\} \leq P(\mathbf{p},\mathbf{x}) \leq \max_i\{p_i\}$$

A 6 *The* ***positivity axiom*** *(Diewert, 1993, p. 8) postulates that*

$$P(\mathbf{p},\mathbf{x}) > 0 \quad \text{and} \quad X(\mathbf{p},\mathbf{x}) > 0, \quad \text{if} \quad \mathbf{p} \gg 0 \quad \text{and} \quad \mathbf{x} \gg 0 \ .$$

Axiom A1 (anonymity) requests that the value of the unilateral price index is exclusively determined by the data pairs (p_i, x_i) and not by other characteristics such as the physical nature of the items considered. Axiom A2 (invariance to re-ordering) postulates that changing the sequence of the items does not affect the value of $P(\mathbf{p},\mathbf{x})$. In the context of intertemporal price measurement, Fisher (1922, p. 63) labelled this axiom as the commodity reversal test. If only one observation existed, then no price aggregation problem would arise. Therefore, Axiom A3 (single observation) postulates for this simplest possible case that $P(\mathbf{p},\mathbf{x})$ should simplify to the observed price p_1. Axiom A4 (uniformity) postulates that for the case of uniform prices ($p_i = p$, $i = 1, 2, ..., N$) the value of $P(\mathbf{p},\mathbf{x})$ should equal this uniform price. By choosing the appropriate quantity units, it is always possible to generate for $P(\mathbf{p},\mathbf{x})$ the value 1. For $N = 1$, Axiom A4 simplifies to Axiom A3. Axiom A5 (mean value) postulates that the value of $P(\mathbf{p},\mathbf{x})$ should always lie between the lowest and the highest of the observed prices. With identical prices, this postulate simplifies to the postulate of Axiom A4 (uniformity). In Axiom A6 (positivity) it is requested that with prices and quantities that are all strictly positive, the value of $P(\mathbf{p},\mathbf{x})$ should also be positive.

A 7 *The* ***product axiom*** *(Eichhorn and Voeller, 1976, p. 75) postulates that*

$$\sum p_i x_i = P(\mathbf{p},\mathbf{x}) \cdot X(\mathbf{p},\mathbf{x}) \; . \tag{1}$$

According to this axiom, multiplication of the unilateral price index with some suitable unilateral quantity index should yield the aggregate value $\sum p_i x_i$ (where $\sum = \sum_{i=1}^{N}$). If one looks at the product axiom in isolation, every unilateral price index satisfies this axiom, because the complementary unilateral quantity index can always be defined as

$$X(\mathbf{p},\mathbf{x}) = \frac{\sum p_i x_i}{P(\mathbf{p},\mathbf{x})} \; .$$

However, the axiom becomes restrictive when it is combined with some other of the listed axioms or with some axioms defined with respect to unilateral quantity indices. For example, the following axiom postulates that an equi-proportional change of all quantities should change the value of $X(\mathbf{p},\mathbf{x})$ by the same proportion.

A 8 *The* ***linear homogeneity (in quantities) axiom*** *(Eichhorn, 1978, p. 145) postulates that*

$$X(\mathbf{p},\lambda\mathbf{x}) = \lambda X(\mathbf{p},\mathbf{x}) \qquad \textit{for all } \lambda > 0 \; .$$

Besides Axioms A6, A7, and A8, also the following axioms have been proposed in the literature.

A 9 *The* ***linear homogeneity (in prices) axiom*** *(Eichhorn and Voeller, 1976, p. 76) postulates that*

$$P(\lambda\mathbf{p},\mathbf{x}) = \lambda P(\mathbf{p},\mathbf{x}) \qquad \textit{for all } \lambda > 0 \; .$$

A 10 *The* ***quantity proportionality axiom*** *(Diewert, 1993, p. 8) postulates that*

$$P(\mathbf{p},\lambda\mathbf{x}) = P(\mathbf{p},\mathbf{x}) \qquad \textit{for all } \lambda > 0 \; .$$

A 11 *The* ***monotonicity axiom*** *(Eichhorn and Voeller, 1976, p. 76) postulates that*

$$P(\mathbf{p},\mathbf{x}) > P(\mathbf{p}^*,\mathbf{x}) \; ,$$

where for all elements $\mathbf{p}$ *and* $\mathbf{p}^*$ *the relation* $p_i^* \geq p_i$ *and for at least one element the strict relation holds.*

A 12 *The* ***weak commensurability axiom*** *(Eichhorn, 1978, p. 145) postulates that*

$$P(\mathbf{p}\lambda,\mathbf{x}/\lambda) = P(\mathbf{p},\mathbf{x}) \qquad \textit{for all } \lambda > 0 \; .$$

A 13 *The* ***strict commensurability axiom*** *(Eichhorn and Voeller, 1976, p. 77) postulates that*

$$P(\mathbf{p}\boldsymbol{\Lambda},\mathbf{x}\boldsymbol{\Lambda}^{-1}) = P(\mathbf{p},\mathbf{x}) \; ,$$

where $\boldsymbol{\Lambda}$ *is an arbitrary* $N \times N$ *diagonal matrix with positive elements* λ_i*.*

Axiom A9 (linear homogeneity in prices) postulates that an equi-proportional change of all prices should change the value of $P(\mathbf{p}, \mathbf{x})$ by the same proportion. According to Axiom A10 (quantity proportionality), $P(\mathbf{p}, \mathbf{x})$ should remain constant when an equi-proportional change of all quantities occurs. Axiom A11 (monotonicity) says that an isolated price increase should always increase the value of $P(\mathbf{p}, \mathbf{x})$. According to Axiom A12 (weak commensurability), a uniform change in the physical units of measurement (e.g., halving all units and, accordingly, doubling all quantities and halving all prices) should not affect the value of $P(\mathbf{p}, \mathbf{x})$. If $P(\mathbf{p}, \mathbf{x})$ violates this axiom, it necessarily violates Axiom A13, because the latter allows for individual changes in the physical units of measurement.

4 Impossibility Theorems

Eichhorn and Voeller (1976, pp. 75-78) show that no unilateral price index $P(\mathbf{p}, \mathbf{x})$ can exist that simultaneously satisfies Axioms A7, A9, A11, and A13. Eichhorn (1978, pp. 144-46) demonstrates that it is impossible for a unilateral price index to simultaneously satisfy Axioms A7, A9, and A12. In Diewert (1993, pp. 7-9) and ILO *et al.* (2004, p. 292) it is shown that no unilateral price index can exist that simultaneously satisfies Axioms A6, A7, A9, A10, and A13. From these impossibility theorems, it has been concluded that the search for a suitable unilateral price index should be abandoned.

It is not difficult to supplement the existing list of impossibility theorems. For example, no unilateral price index formula can exist that simultaneously satisfies Axioms A9, A10, and A12, because from A9 and then A10 the following relationship follows:

$$P(\mathbf{p}/\lambda, \mathbf{x}\lambda) = (1/\lambda)P(\mathbf{p}, \mathbf{x}\lambda) = (1/\lambda)P(\mathbf{p}, \mathbf{x}) \ ,$$

which contradicts Axiom A12. Another impossibility finding involves Axioms A3 and A12, and therefore also Axioms A4 and A12. If one applies the axioms in the sequence A3-A12-A3, then they generate the contradiction

$$p = P(\mathbf{p}, \mathbf{x}) = \lambda P(\mathbf{p}/\lambda, \mathbf{x}\lambda) = p/\lambda \ .$$

These additional impossibility theorems involve particularly few and weak axioms. In this sense, they are stronger results than the impossibility theorems that hitherto have been proposed in the literature. Therefore, they appear to re-inforce the view that no meaningful unilateral price index can exist.

5 Re-Considering the Theorems

All of the listed impossibility theorems rely on the postulate that a suitable unilateral price index must satisfy the weak commensurability axiom A12 or even its stricter version A13. However, these two axioms, though indispensable in the context of intertemporal price measurement, are misplaced in the context of price level measurement, that is, for unilateral price indices $P(\mathbf{p}, \mathbf{x})$.

A simple intuition for this statement is provided by the last of the impossibility theorems. Suppose that only one observation exists: (p_1, x_1). Therefore, no aggregation or index problem arises. The observed price p_1 represents the obvious measure of the prevailing price level: $P(\mathbf{p}, \mathbf{x}) = p_1$ (single observation axiom A3). In this simple case, the aggregate value is $p_1 x_1$. Suppose that the physical units of measurement are halved. If $P(\mathbf{p}, \mathbf{x})$ satisfied the weak or strict commensurability axiom, one would get $P(\mathbf{p}/2, \mathbf{x}2) = P(\mathbf{p}, \mathbf{x})$. Invoking A3, this is equivalent with $\tilde{p}_1 = p_1$. Halving the physical units of measurement also implies that the new quantity is $\tilde{x}_1 = 2x_1$. As a consequence, the new aggregate value would no longer be $p_1 x_1$ but $\tilde{p}_1 \tilde{x}_1 = 2p_1 x_1$, even though the currency has not been changed. In other words, postulating that the unilateral price index must satisfy the weak or strict commensurability axiom is postulating that the aggregate value changes as the physical units of measurement are changed. However, nobody would seriously want the aggregate value to change. It can be concluded from this argument that a meaningful unilateral price index $P(\mathbf{p}, \mathbf{x})$ must violate the weak and strict commensurability axiom. For example, when the physical units of measurement are halved, the new price should be $\tilde{p}_1 = p_1/2$.

In the previous paragraph it was demonstrated that in the context of a single observation the weak and strict commensurability axioms are inappropriate. A similar argument can be used when several observations (quantities and prices) are considered that all relate to the same homogeneous item. The aggregate value of the homogeneous item is $\sum p_i x_i$. This aggregate value can be decomposed as in Equation (1). Since all observations relate to the same homogeneous item, the quantity component is

$$X(\mathbf{p}, \mathbf{x}) = \sum x_i \tag{2}$$

and the price component $P(\mathbf{p}, \mathbf{x})$ can be interpreted as an average price level. From Equations (1) and (2) one obtains for the unilateral price index the expression

$$P_{UV} := P(\mathbf{p}, \mathbf{x}) = \frac{\sum p_i x_i}{\sum x_i} \; . \tag{3}$$

This is the unit value formula proposed by Segnitz (1870, p. 184) and it is widely considered as the natural index formula for aggregating the observed prices of some homogeneous item. Note that the aggregate value obtained by multiplying the two formulas (2) and (3) remains unaffected when the physical units of measurement are changed. At the same time both formulas violate the weak and therefore also the strict commensurability axiom. For (3) one obtains

$$P_{UV}(\mathbf{p}/\lambda, \mathbf{x}\lambda) = \frac{\sum (p_i/\lambda) x_i \lambda}{\sum x_i \lambda} = \frac{1}{\lambda} \frac{\sum p_i x_i}{\sum x_i} = \frac{1}{\lambda} P_{UV}(\mathbf{p}, \mathbf{x}) \; .$$

All this re-inforces the line of reasoning used in the context of a single observation: A meaningful unilateral price index must violate the weak and strict commensurability axiom.

To summarize, there is unanimity that for the observations of a homogeneous item the unit value formula is the obvious measure of the prevailing price level. Therefore, it is the adequate unilateral price index. In contrast to the aggregate value ($\sum p_i x_i$), the values of this unilateral price index and the complementary unilateral quantity index ($\sum x_i$) strongly depend on the physical units of measurement. In other words, they violate the weak and strict commensurability axiom. This is a desirable property.

Between the cases of a perfectly homogeneous item and strongly heterogeneous items there is a continuum of cases. Of course, looking at a case with a trace of heterogeneity, the unit value formula (3) would no longer be the fully satisfactory unilateral price index. However, there is no reason why for this new case the values of the applied unilateral price and quantity index should suddenly remain constant when the physical units of measurement are changed. Also with increasingly heterogeneous items, the argument remains valid. It is impossible to define a specific degree of heterogeneity from which on one could suddenly regard the commensurability axioms as making sense.

6 Concluding Remarks

In economics there is a widely held belief that an economy's general intertemporal price change between two periods of time should not be measured as the change in the general price levels of the two periods, that is, as the ratio of the respective unilateral price indices. The existing formal justifications supporting this belief rest on the postulate that the unilateral price indices should be invariant with respect to the units of measurement (commensurability axiom). In the present study it was argued that for a unilateral price index this postulate is misplaced. A meaningful unilateral price index must depend on the units of measurement.

Does this mean that an economy's intertemporal price change can be measured as the ratio of two unilateral price indices? This conclusion would be premature. This study has merely demonstrated that the existing formal objections against the development of unilateral price indices are not convincing. Other new objections might turn out to be more convincing. Furthermore, a ratio of two unilateral price indices represents a bilateral price index. This changes the axiomatic perspective. Such an index must satisfy axioms specifically designed for the case of bilateral price indices. For example, any meaningful bilateral price index (in contrast to a meaningful unilateral price index) must satisfy the strict commensurability axiom. In Auer (2008) a new class of bilateral price indices is developed that can be viewed as ratios of two unilateral price indices. This class is denoted as the family of generalized unit value indices. The study also provides an axiomatic analysis of this family of price indices.

7 References

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