Reinsdorf-Balk transformation and additive decomposition of indexes

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Abstract

Recently Reinsdorf(1997), Reinsdorf, Diewert and Ehemann (2002) and Balk (2004) have made a remarkable contribution to the index number theory. They showed that the two representative classes of index numbers such as arithmetic mean indexes and geometric mean indexes are interchangeable. In this paper we refine these ideas and provide some applications with numerical illustrations.

I. Introduction

We define an arithmetic mean quantity index as (1a) and a geometric mean quantity index as (1b), where weight function s_i and w_i are some expenditure shares of i[th] item. The arithmetic mean index includes such indexes as the Laspeyres index the Paasche index, and more generally the Lowe index.

$$Q = \sum_{i=1}^{N} s_i \frac{q_{i,1}}{q_{i,0}} , \quad \sum_{i=1}^{N} s_i = 1 \tag{1a}$$

$$Q' = \prod_{i=1}^{N} \left(\frac{q_{i,1}}{q_{i,0}} \right)^{w_i} , \quad \sum_{i=1}^{N} w_i = 1 \tag{1b}$$

The arithmetic mean index (1a) implies an additive decomposition in percent-change as (2a), while the geometric mean index (1b) does an additive decomposition in log-change as (2b). Balk(2004) called (2b) as a multiplicative decomposition based on the equivalence between (1b) and (2b), while Kohli(2007) called (1b) as a multiplicative decomposition.

$$Q - 1 = \sum_{i=1}^{N} s_i \left(\frac{q_{i,1}}{q_{i,0}} - 1 \right) , \quad \sum_{i=1}^{N} s_i = 1 \tag{2a}$$

$$\ln Q' = \sum_{i=1}^{N} w_i \ln \frac{q_{i,1}}{q_{i,0}} , \quad \sum_{i=1}^{N} w_i = 1 \tag{2b}$$

The corresponding terms of (2a) and (2b) have well-known approximation relation. The Taylor expansion of $\ln x$ around unity (3) shows that the corresponding values are similar when $q_{i,1} \cong q_{i,0}$ and that log-changes tend to be smaller than percent-changes.

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \cdots \tag{3}$$

Theil(1978, pp.188-194) and Törnqvist et al.(1985) preferred the log-change to the more common percent-change on the ground that the former is more scientific. We, however, are in favor of the percent-change, since the log-change loses its practical meaning when it diverges from the percent-change that is firmly based on common sense.

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Most national statistical agencies had used arithmetic mean indexes, such as a Laspeyres index or a Paasche index. Therefore the additive decomposition of percent-change (2a) had prevailed. However, as the national statistical agencies of the U.S. and Canada substitute the Laspeyres index with Fisher chain index for compiling the real GDP, two problems arised with respect to the additive decomposition of its growth rates. The first is about how to define the additive decomposition of Fisher index that is not arithmetic mean index nor a geometric mean index. The second is how to do the additive decomposition of a chain index.

Regarding the first problem there have appeared several methods to transform the Fisher index to either form of (1a) or (1b). Diewert(2002), Dumagan(2002), Ehemann et al(2002), Hallerbach(2005) provided their own version of Fisher index as an arithmetic mean index or equivalently the additive decomposition in percent-change. However, those of Dumagan(2002), Ehemann et al(2002) were turned out to be re-discoveries of that of van IJzeren(1952) not so well known. The version of van IJzeren is known as the best in the axiomatic approach to date. On the other hand, Reinsdorf(1997), Reinsdorf et al(2002), Balk(2004) provides a transformation of Fisher index to a geometric mean index or equivalently an additive decomposition in log-change.

On the other hand, there have been few studies that focused on the second problem of an additive decomposition of chain indexes in percent-changes. Since the chain index of a geometric mean index is also a geometric mean index, its additive decomposition in log-change can be easily defined. However, we note that for chain indexes with two or more time periods an additive decomposition in log-change does not have so much practical meaning since it deviates from the percent-change.

This paper reconsiders the two problems using the ideas in Reinsdorf(1997), Reinsdorf et al(2002), Balk(2004). In section II, a transformation between an arithmetic mean index and a geometric mean index in Reinsdorf(1997), Reinsdorf et al(2002), Balk(2004) is provided with more useful form which makes explicit the linkage between the two classes of indexes. In section III, we rewrite the additive decomposition in log-changes (or multiplicative decomposition) of Fisher index using our form. We prove that the arithmetic mean version (or fixed basket version or Lowe index version) of Fisher index mentioned in Reinsdorf et al(2002, p.58) differ algebraically from that of van IJzeren, though almost identical numerically. We also provide a new geometric mean version of Fisher index unnoticed in the literature. In section IV, we define an additive decomposition in percent-change for chain index, which can show the long-run trend of contributions. In section V, we provide numerical illustrations using the US. BEA data set and concluding remarks follows.

II. Reinsdorf-Balk Transformation

Reinsdorf(1997), Reinsdorf et.al.(2002), Balk(2004) showed that an arithmetic mean index (1a) and a geometric mean index (1b) can be transformed to each other. In their procedure, the logarithmic mean plays a central role. The logarithmic mean, introduced to the economic literature by Törnqvist in 1935 (Törnqvist et al., 1985), Vartia(1976) and Sato(1976), is defined as (4) and locates between an arithmetic mean and a geometric mean.

$$L(x,y) = \frac{x-y}{\ln x/y} \ \text{ if } \ x \neq y, \qquad L(x,y) = x = y \ \text{ if } \ x = y \tag{4}$$

Reinsdorf(1997) and Reinsdorf et al.(2002) showed that a geometric mean index can be transformed to a fixed basket index (in the context of price index) or an arithmetic mean index. They also showed that an arithmetic mean index could be transformed to a geometric mean index. Balk(2004) focused on the transformation in the latter direction. He used an ingenious proof by which he defined the ideal log-change index, i.e, Sato-Vartia index in Balk(1999). This geometric mean versions of fixed basket indexes are superior to the one by Vartia(1976), Barnett et al.(2003) in the sense of homogeneity property.

We call the essential idea in Balk(1999, 2004) as Balk's identity. We refine the identity by providing a more useful form by incorporating Reinsdorf(1997), Reinsdorf et al.(2002).

<u>Lemma : Balk's identity</u>

An arithmetic mean index Q (or the Lowe index) and a geometric mean index Q', are identical if the weights of a geometric mean index is defined as (5b), where $L(x,y)$ is a logarithmic mean.

$$Q = \sum_{i=1}^{N} s_i \frac{q_{i,1}}{q_{i,0}}, \ s_i = \frac{q_{i,1} p_i}{\sum_{i=1}^{N} q_{i,0} p_i}, \ Q' = \prod_{i=1}^{N} \left(\frac{q_{i,1}}{q_{i,0}} \right)^{w_i} \tag{5a}$$

$$w_i = \frac{p_i L(q_{i,1}, q_{i,0} Q)}{\sum_{k=1}^{N} p_k L(q_{k,1}, q_{k,0} Q)} \tag{5b}$$

(proof)

Define $s_i = s_{i,0} = \frac{q_{i,0}\, p_i}{\sum_{k=1}^{N} q_{k,0}\, p_k}$, $s_{i,1} = \frac{q_{i,1}\, p_i}{\sum_{k=1}^{N} q_{k,1}\, p_k}$ then $\sum_{i=1}^{N} (s_{i,1} - s_{i,0}) = 0$ is an identity. By the definition of logarithmic mean, the following is an identity.

$$\sum_{i=1}^{N} L(s_{i,1}, s_{i,0}) \ln \frac{s_{i,1}}{s_{i,0}} = 0 \tag{6}$$

If all $s_{i,1}$ and $s_{i,0}$ are non-zero and have the same sign, then the following is an identity.

$$\ln\frac{s_{i,1}}{s_{i,0}} = \ln\frac{q_{i,1}p_i}{q_{i,0}p_i} - \ln\frac{\sum_{k-1}^{N} q_{k,1}p_k}{\sum_{k-1}^{N} q_{k,0}p_k} = \ln\frac{q_{i,1}}{q_{i,0}} - \ln Q \tag{7}$$

Inserting into (6) the most right hand side of (7), the following is an identity.

$$\sum_{i=1}^{N} L(s_{i,1}, s_{i,0})\left(\ln\frac{q_{i,1}}{q_{i,0}} - \ln Q\right) = 0 \quad \Leftrightarrow \quad \ln Q = \sum_{i=1}^{N}\frac{L(s_{i,1}, s_{i,0})}{\sum_{k=1}^{N} L(s_{k,1}, s_{k,0})}\ln\frac{q_{i,1}}{q_{i,0}} =$$

$$\sum_{i=1}^{N}\frac{p_i L(q_{i,1}, q_{i,0})}{\sum_{k=1}^{N} p_k L(q_{k,1}, q_{k,0}Q)}\ln\frac{q_{i,1}}{q_{i,0}} = \sum_{i=1}^{N} w_i \ln\frac{q_{i,1}}{q_{i,0}} \tag{8}$$

Since the most right hand side of (8) is $\ln Q'$, therefore $Q = Q'$ holds. □

We define the weights of a geometric mean index as (5b) instead of several alternative forms in Balk(2004) and Reinsdorf et al(2002) because it is a form of mean quantity familiar in the index number theory. It also makes explicit a linkage between a geometric mean index and an arithmetic mean index. Since an arithmetic mean index Q is homogeneous of degree zero in price, the form (5b) implies price function (9) used in Reinsdorf et al(2002) for the transformation from a geometric mean index to an arithmetic mean index.

$$p_i = \frac{w_i}{L(q_{i,1}, q_{i,0}Q)} \tag{9}$$

Reinsdorf et al(2002) showed that a geometric mean index can be transformed to an arithmetic mean index. The proof, however, is rather complicated one using tricky algebraic manipulations. Instead of it we prove it as a corollary of Balk's identity and named it as Reinsdorf's identity.

<u>Corollary : Reinsdorf's identity.</u>

Q is an arithmetic mean index and Q' is a geometric mean index as defined in (10a). Then $Q = Q'$ is an identity if price function for an arithmetic mean index is defined as (10b), where $L(x,y)$ is a logarithmic mean.

$$Q = \frac{\sum_{i=1}^{N} q_{i,1}p_i}{\sum_{i=1}^{N} q_{i,0}p_i} \equiv \sum_{i=1}^{N} s_i \frac{q_{i,1}}{q_{i,0}} \;\; , \;\; Q' = \prod_{i=1}^{N}\left(\frac{q_{i,1}}{q_{i,0}}\right)^{w_i} \tag{10a}$$

$$p_i = \frac{w_i}{L(q_{i,1}, q_{i,0}Q)} \tag{10b}$$

(proof)

An arithmetic mean index Q can be written as (11) by Balk's identity.

$$\ln Q = \sum_{i=1}^{N} \frac{p_i L(q_{i,1}, q_{i,0}Q)}{\sum_{k=1}^{N} p_k L(q_{k,1}, q_{k,0}Q)} \ln \frac{q_{i,1}}{q_{i,0}} \tag{11}$$

Substituting the p_i in (11) with the right hand side of (10b), we obtain the identity $Q = Q'$.

$$\ln Q = \sum_{i=1}^{N} \frac{p_i L(q_{i,1}, q_{i,0}Q)}{\sum_{k=1}^{N} p_k L(q_{k,1}, q_{k,0}Q)} \ln \frac{q_{i,1}}{q_{i,0}} = \sum_{i=1}^{N} \frac{w_i}{\sum_{k=1}^{N} w_k} \ln \frac{q_{i,1}}{q_{i,0}} = \sum_{i=1}^{N} w_i \ln \frac{q_{i,1}}{q_{i,0}} = \ln Q' \tag{12}$$

□

If we integrate the two identities, an arithmetic mean index and a geometric mean index can be transformed to each other by the following formulas (13a) and (13b). The former is a transformation from an arithmetic mean index to a geometric mean index and the latter is a transformation in reverse direction.

Theorem: Reinsdorf-Balk transformation

$$Q = \frac{\sum_{i=1}^{N} q_{i,1} p_i}{\sum_{i=1}^{N} q_{i,0} p_i} \Rightarrow \ln Q = \sum_{i=1}^{N} \frac{p_i L(q_{i,1}, q_{i,0}Q)}{\sum_{k=1}^{N} p_k L(q_{k,1}, q_{k,0}Q)} \ln \frac{q_{i,1}}{q_{i,0}} \tag{13a}$$

$$\ln Q = \sum_{i=1}^{N} w_i \ln \frac{q_{i,1}}{q_{i,0}} \Rightarrow Q = \frac{\sum_{i=1}^{N} q_{i,1} \frac{w_i}{L(q_{i,1}, q_{i,0}Q)}}{\sum_{i=1}^{N} q_{i,0} \frac{w_i}{L(q_{i,1}, q_{i,0}Q)}} \tag{13b}$$

(proof)

(13a) follows from Balk's identity and (13b) follows from Reinsdorf's identity. □

III. Additive decompositions of Fisher index

The ideal Fisher index is neither an arithmetic mean index nor a geometric mean index. But Van IJzeren(1952) showed that Fisher index could be transformed to an arithmetic mean index, while Reinsdorf(1997), Reinsdorf et al(2002), and Balk(2004) showed that it can be represented by a geometric mean index. In addition to these two conventional additive decompositions in literature, we provide two new additive decompositions.

(1) Conventional additive decompositions of Fisher index

As is well-known, van IJzeren(1952) provided an arithmetic mean version of Fisher index (14), where P^F is Fisher price index. Recently, Dumagan(2002), Ehemann et al(2002) re-discovered the same result.

$$Q^F = \frac{\sum_{i=1}^{N} q_{i,1}(p_{i,1} + p_{i,0}P^F)}{\sum_{i=1}^{N} q_{i,0}(p_{i,1} + p_{i,0}P^F)} = \frac{\sum_{i=1}^{N} q_{i,1}P_i^v}{\sum_{i=1}^{N} q_{i,0}P_i^v} \tag{14}$$

According to Balk's identity, an arithmetic mean indexes such as Laspeyres index Q^L and Paasche index Q^P can be transformed to geometric mean indexes as (15). Vartia(1976) first provided such kind of transformations to Laspeyres index and Paasche index but they are inferior to (15), since their weights are not sum to unity.

$$\ln Q^L = \sum_{i=1}^{N} w_i^{\,L} \ln\frac{q_{i,1}}{q_{i,0}},\; w_i^{\,L} = \frac{p_{i,0}L(q_{i,1}, q_{i,0}Q^L)}{\sum_{k=1}^{N} p_{k,0}L(q_{k,1}, q_{k,0}Q^L)} \tag{15a}$$

$$\ln Q^P = \sum_{i=1}^{N} w_i^{\,P} \ln\frac{q_{i,1}}{q_{i,0}},\; w_i^{\,P} = \frac{p_{i,1}L(q_{i,1}, q_{i,0}Q^P)}{\sum_{k=1}^{N} p_{k,1}L(q_{k,1}, q_{k,0}Q^P)} \tag{15b}$$

The Fisher index $Q^F = \sqrt{Q^L \times Q^P}$ can be transformed to a geometric mean index as follows using (15).

$$\ln Q^F = \frac{1}{2}\left(\ln Q^L + \ln Q^P\right) = \sum_{i=1}^{N} \frac{w_i^{\,L} + w_i^{\,P}}{2} \ln\frac{q_{i,1}}{q_{i,0}} \equiv \sum_{i=1}^{N} w_i^{\,R-B} \ln\frac{q_{i,1}}{q_{i,0}} \tag{16}$$

(2) New additive decompositions of Fisher index

It has been unnoticed that there is one more geometric mean version of Fisher index in addition to (16). It is almost trivial to obtain the following geometric mean Fisher index (17) from van IJzeren's version of Fisher index (14) using Balk's identity. This version is transformed to van IJzeren's Fisher index by Reinsdorf identity.

$$\ln Q^F = \sum_{i=1}^{N} \frac{(p_{i,1} + p_{i,0}P^F)L(q_{i,1}, q_{i,0}Q^F)}{\sum_{k=1}^{N}(p_{k,1} + p_{k,0}P^F)L(q_{k,1}, q_{k,0}Q^F)} \ln\frac{q_{i,1}}{q_{i,0}} = \sum_{i=1}^{N} \frac{p_i^{\,v} L(q_{i,1}, q_{i,0}Q^F)}{\sum_{k=1}^{N} p_k^{\,v} L(q_{k,1}, q_{k,0}Q^F)} \ln\frac{q_{i,1}}{q_{i,0}} \tag{17}$$

In addition to van IJzeren's arithmetic mean version of Fisher index (14), we have another arithmetic mean version of Fisher index (18) by applying Reinsdorf's identity to (16),

$$Q^F = \frac{\sum_{i=1}^{N} q_{i,1} \frac{w_i^{R-B}}{L(q_{i,1}, q_{i,0}Q^F)}}{\sum_{i=1}^{N} q_{i,0} \frac{w_i^{R-B}}{L(q_{i,1}, q_{i,0}Q^F)}} = \frac{\sum_{i=1}^{N} q_{i,1} P_i^R}{\sum_{i=1}^{N} q_{i,0} P_i^R} \tag{18}$$

Though Reinsdorf et al(2002, p.58) first derived (18), they just wrote that it is "numerically equivalent" to (14) observing that their values are very close. We show that they are different algebraically, that is $p_i^{\,v} \neq p_i^{\,R}$. From the structure of (19), we see that $p_i^{\,R} = a_i p_{i,1} + b_i p_{i,0}$, where a_i and b_i are independent positive numbers. This shows that $p_i^{\,R} = a_i p_{i,1} + b_i p_{i,0} \neq p_{i,1} + p_{i,0} P^F = p_i^{\,v}$.

$$p_i^{\,R} = \frac{w_i^{R-B}}{L(q_{i,1}, q_{i,0}Q^F)} = \frac{0.5\,L(q_{i,1}, q_{i,0}Q^P)}{\sum_{k=1}^{N} p_{k,1} L(q_{k,1}, q_{k,0}Q^P) L(q_{i,1}, q_{i,0}Q^F)}\, p_{i,1} + \frac{0.5\,L(q_{i,1}, q_{i,0}Q^L)}{\sum_{k=1}^{N} p_{k,0} L(q_{k,1}, q_{k,0}Q^L) L(q_{i,1}, q_{i,0}Q^F)}\, p_{i,0} \tag{19}$$

IV. Additive decomposition of chain indexes

For a chain index with geometric mean index as a link factor, its additive decomposition in log-change (or multiplicative decomposition) is rather straightforward as following. Even if the link factor is an arithmetic mean index such as the Laspeyres index, we can always re-write it as a geometric mean index by Balk's identity.

$$\ln Q^c{}_{0,T} = \ln \prod_{t=1}^{T} Q_{t-1,t} = \sum_{t=1}^{T} \ln Q_{t-1,t} = \sum_{t=1}^{T}\sum_{i=1}^{N} w_{i,t-1,t} \ln \frac{q_{i,t}}{q_{i,t-1}} = \sum_{i=1}^{N}\sum_{t=1}^{T} w_{i,t-1,t} \ln \frac{q_{i,t}}{q_{i,t-1}} \tag{20}$$

In the above log-change form of the chain index, the contribution of i^{th} source during period [0,T] can be defined as (21).

$$\sum_{t=1}^{T} w_{i,t-1,t} \ln \frac{q_{i,t}}{q_{i,t-1}} = w^c{}_{i,0,T} \ln \frac{q_{i,T}}{q_{i,0}}, \qquad w^c{}_{i,0,T} \equiv \left(\sum_{t=1}^{T} w_{i,t-1,t} \ln \frac{q_{i,t}}{q_{i,t-1}} \right) \Bigg/ \ln \frac{q_{i,T}}{q_{i,0}} \tag{21}$$

Plugging (21) into (20), we have an additive decomposition in log-change as (22). This additive decomposition in log-change, however, is not so useful, because log-changes deviate from percent-changes since $Q^c{}_{0,T}$, $\frac{q_{i,T}}{q_{i,0}}$ in (22) is not so close to unity in many cases.

$$\ln Q^c{}_{0,T} = \sum_{i=1}^{N} w^c{}_{i,0,T} \ln \frac{q_{i,T}}{q_{i,0}} \tag{22}$$

Though we have Reinsdorf's identity that can transformation a multiplicative decomposition (or additive decomposition in log-change) to an additive decomposition, the multiplicative decomposition (22) is not in standard form, since the sum of weights $w^c{}_{i,0,T}$ is not unity in general. As is familiar, we normalize the weights by a scale factor, retaining the identity as (23), which is a standard additive decomposition in log-change.

$$\ln Q_{0,T}{}^c = \sum_{i=1}^{N} \frac{w_{i,0,T}{}^c}{\sum_{k=1}^{N} w_{k,0,T}{}^c} \sum_{k=1}^{N} w_{k,0,T}{}^c \ln \frac{q_{i,T}}{q_{i,0}} = \sum_{i=1}^{N} \frac{w_{i,0,T}{}^c}{\sum_{k=1}^{N} w_{k,0,T}{}^c} \ln \frac{q_{i,T}{}^{\lambda}}{q_{i,0}{}^{\lambda}}, \quad \lambda \equiv \sum_{i=1}^{N} w_{i,0,T}{}^c \tag{23}$$

Now $Q^c{}_{0,T}$ can be transformed to an arithmetic mean index by Reinsdorf's identity and hence resulting in an additive decomposition in percent-change as (24)

$$Q^c{}_{0,T} - 1 = \frac{\displaystyle\sum_{i=1}^{N} \frac{w^c{}_{i,0,T}}{\lambda L\left(q_{i,T}{}^{\lambda}, q_{i,0}{}^{\lambda} Q^c{}_{0,T}\right)} q_{i,T}{}^{\lambda}}{\displaystyle\sum_{i=1}^{N} \frac{w^c{}_{i,0,T}}{\lambda L\left(q_{i,T}{}^{\lambda}, q_{i,0}{}^{\lambda} Q^c{}_{0,T}\right)} q_{i,0}{}^{\lambda}} - 1 = \sum_{i=1}^{N} \frac{\frac{w^c{}_{i,0,T}}{L\left(q_{i,T}{}^{\lambda}, q_{i,0}{}^{\lambda} Q^c{}_{0,T}\right)} q_{i,0}{}^{\lambda}}{\displaystyle\sum_{k=1}^{N} \frac{w^c{}_{k,0,T}}{L\left(q_{k,T}{}^{\lambda}, q_{k,0}{}^{\lambda} Q^c{}_{0,T}\right)} q_{k,0}{}^{\lambda}} \left(\frac{q_{i,T}{}^{\lambda} - q_{i,0}{}^{\lambda}}{q_{i,0}{}^{\lambda}} \right) \tag{24}$$

In the process the normalization may seem to be arbitrary. But this procedure is conjectured to be robust since the sum of weights $w^c{}_{i,0,T}$ in (21) tends to unity as time T increases. We will show it in a numerical example later.

V. Numerical Illustrations: additive decomposition of Fisher chain index

Kohli(2007) applied the Reinsdorf-Balk transformations to the decomposition of real GDP growth rate using the U.S. BEA data set. We illustrate some analytical statements in this paper using the same data set.

In Table 1, we can see two GDPs, column (a) by direct Fisher index and column (b) by Fisher chain index with base year 1980. For the two Fisher indexes, we contrast growth rates by two measures; by percent-change units and by log-change units.

Table 1. Real GDP by Fisher indexes

	Fisher direct index (base=previous year)			Fisher chain index (base=1980)		
	GDP (a)	%-change (b)	log-change (c)	GDP (d)	%-change (e)	log-change (f)
1981	1.0244	2.44	2.41	1.0244	2.44	2.41
1982	0.9798	-2.02	-2.04	1.0037	0.37	0.37
1983	1.0433	4.33	4.24	1.0472	4.72	4.61
1984	1.0727	7.27	7.01	1.1233	12.33	11.62
1985	1.0385	3.85	3.78	1.1665	16.65	15.40
1986	1.0342	3.42	3.36	1.2064	20.64	18.76
1987	1.0340	3.40	3.34	1.2474	24.74	22.10
1988	1.0417	4.17	4.09	1.2994	29.94	26.19
1989	1.0351	3.51	3.45	1.3450	34.50	29.64
1990	1.0176	1.76	1.74	1.3687	36.87	31.38
1991	0.9953	-0.47	-0.47	1.3622	36.22	30.91
1992	1.0305	3.05	3.00	1.4038	40.38	33.92
1993	1.0265	2.65	2.62	1.4410	44.10	36.54
1994	1.0403	4.03	3.95	1.4992	49.92	40.49
1995	1.0267	2.67	2.63	1.5392	53.92	43.12
1996	1.0357	3.57	3.51	1.5942	59.42	46.64
1997	1.0443	4.43	4.34	1.6648	66.48	50.97
1998	1.0428	4.28	4.19	1.7361	73.61	55.16
1999	1.0411	4.11	4.03	1.8075	80.75	59.19
2000	1.0375	3.75	3.68	1.8753	87.53	62.88
2001	1.0025	0.25	0.25	1.8800	88.00	63.13

Notes : (a) $Q_{t-1,t}{}^{F}$, (b) $\left(Q_{t-1,t}{}^{F}-1\right)\times 100$, (c) $\ln Q_{t-1,t}{}^{F}\times 100$,

(d) $Q_{1980,t}{}^{F,c}=\prod_{k=1981}^{t} Q_{k-1,k}$, (e) $\left(Q_{1980,t}{}^{F,c}-1\right)\times 100$, (f) $\ln Q_{1980,t}{}^{F,c}\times 100$

In Table 1, we can see that the discrepancies between the growth rates measured by percent-changes and log-changes are overall very small in direct Fisher indexes, while those between growth rates measured by percent-changes and log-changes are growing as the time span increases in the Fisher chain indexes. We think the growth rates in log-changes loses its practical meaning when it comes to chain indexes. Thus for the Fisher chain indexes we only consider an additive decomposition in percent-changes.

For the Fisher direct indexes, we compare the two additive decompositions of real GDP annual growth rates by its five components in Table 2. Overall, there are no significant difference between the two measures. We use the van IJzeren form of Fisher index for the additive decomposition in percent-changes in the left hand side of the table and use the Reinsdorf-Balk form of Fisher index for the additive decomposition in log-changes (or multiplicative decomposition) in the right hand side of the table. The additive decomposition in percent-changes is exactly the same with Table 1 in Kohli(1971).

Table 2. Additive decomposition of annual GDP growth rates

	%-change						log-change					
	(a)	(b)	(c)	(d)	(e)	GDP	(a)	(b)	(c)	(d)	(e)	GDP
1981	0.85	1.57	0.18	0.11	-0.27	2.44	0.84	1.55	0.18	0.11	-0.27	2.41
1982	0.76	-2.55	0.31	-0.67	0.12	-2.02	0.76	-2.57	0.32	-0.67	0.12	-2.04
1983	3.49	1.48	0.70	-0.21	-1.13	4.33	3.41	1.44	0.69	-0.20	-1.11	4.24
1984	3.49	4.62	0.73	0.65	-2.21	7.27	3.37	4.46	0.70	0.63	-2.14	7.01
1985	3.15	-0.17	1.31	0.20	-0.64	3.85	3.09	-0.17	1.29	0.20	-0.63	3.78
1986	2.70	-0.12	1.13	0.52	-0.82	3.42	2.66	-0.11	1.11	0.51	-0.81	3.36
1987	2.17	0.42	0.63	0.81	-0.63	3.40	2.13	0.41	0.62	0.79	-0.61	3.34
1988	2.65	0.44	0.24	1.25	-0.41	4.17	2.60	0.43	0.24	1.22	-0.40	4.09
1989	1.76	0.60	0.56	1.02	-0.43	3.51	1.73	0.59	0.55	1.01	-0.42	3.45
1990	1.21	-0.49	0.65	0.80	-0.41	1.76	1.20	-0.49	0.65	0.79	-0.41	1.74
1991	-0.12	-1.26	0.24	0.62	0.05	-0.47	-0.12	-1.26	0.24	0.62	0.05	-0.47
1992	1.91	1.12	0.10	0.61	-0.68	3.05	1.88	1.10	0.10	0.60	-0.67	3.00
1993	2.24	1.18	-0.16	0.33	-0.94	2.65	2.21	1.17	-0.16	0.33	-0.93	2.62
1994	2.53	1.89	0.02	0.88	-1.29	4.03	2.48	1.86	0.02	0.86	-1.26	3.95
1995	2.00	0.47	0.08	1.06	-0.95	2.67	1.98	0.46	0.08	1.04	-0.93	2.63
1996	2.14	1.37	0.20	0.89	-1.03	3.57	2.11	1.34	0.20	0.87	-1.02	3.51
1997	2.39	1.91	0.43	1.35	-1.64	4.43	2.33	1.87	0.42	1.32	-1.60	4.34
1998	3.18	1.95	0.34	0.24	-1.44	4.28	3.12	1.91	0.34	0.24	-1.41	4.19
1999	3.30	1.14	0.68	0.37	-1.38	4.11	3.23	1.12	0.67	0.36	-1.35	4.03
2000	2.94	1.08	0.49	1.04	-1.79	3.75	2.88	1.06	0.48	1.02	-1.75	3.68
2001	1.67	-1.90	0.65	-0.59	0.42	0.25	1.67	-1.90	0.65	-0.59	0.42	0.25

Notes : (a) consumption, (b) investment, (c) government, (d) export, (e) import

We also compared the numerical similarities between respective alternatives; that is, the left hand side of the Table 2 with the Reinsdorf form of the additive decomposition in percent-changes as (25a), and the right hand side of the table with the van IJzeren form additive decomposition in log-changes as (25b). According to our numerical experiments they are extremely similar, exactly the same until eighth decimal points in every values. They are much closer than we expected.

$$Q^F - 1 = \sum_{i=1}^{5} \frac{q_{i,0}\, p_i^{\,R}}{\sum_{k=1}^{5} q_{k,0}\, p_i^{\,R}} \left(\frac{q_{i,1}}{q_{i,0}} - 1 \right), \qquad p_i^{\,R} \equiv \frac{w_i^{\,R-B}}{L\left(q_{i,1}, q_{i,0}\, Q^F\right)} \tag{25a}$$

$$\ln Q^F = \sum_{i=1}^{N} w_i^{\,v} \ln \frac{q_{i,1}}{q_{i,0}}, \qquad w_i^{\,v} \equiv \frac{(p_{i,1} + p_{i,0} P^F) L(q_{i,1}, q_{i,0} Q^F)}{\sum_{k=1}^{N} (p_{k,1} + p_{k,0} P^F) L(q_{k,1}, q_{k,0} Q^F)} \tag{25b}$$

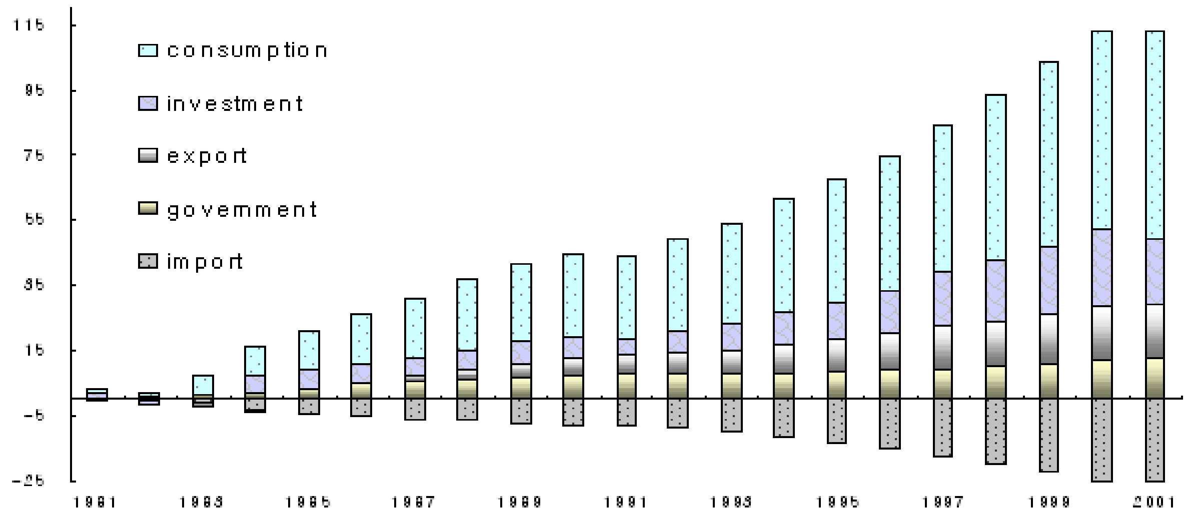
In the following Table 3 we provide an additive decomposition in percent-change for the Fisher chain index. As we mentioned, we think the additive decomposition in log-changes does not have practical meaning for chain indexes. As we describe in section IV, we first derive a standard geometric mean version of Fisher index by Reinsdorf-Balk with required normalization. Applying the Reinsdorf's identity to the chain indexes, we obtain an additive decomposition in percent-changes. As we mentioned, it can be seen that the normalizing factor in the right most column is close to unity and seems to tends to unity as time span increases. This analysis can also be represented as Figure 1, which shows long-run trends of relative contributions by five sources to the cumulative GDP growth rates.

Table 3. Real GDP by Fisher chain index

	Additive decomposition of %-change						λ
	(a)	(b)	(c)	(d)	(e)	GDP	
1981	0.85	1.57	0.18	0.11	-0.27	2.44	1.00000
1982	1.60	-1.02	0.50	-0.56	-0.15	0.37	0.97497
1983	5.13	0.44	1.21	-0.78	-1.28	4.72	0.99281
1984	8.89	5.19	2.00	-0.15	-3.60	12.33	1.13596
1985	12.41	5.11	3.43	0.06	-4.36	16.65	0.94750
1986	15.56	5.07	4.72	0.63	-5.34	20.64	0.98347
1987	18.23	5.62	5.49	1.52	-6.13	24.74	0.99091
1988	21.59	6.24	5.88	2.96	-6.73	29.94	0.99723
1989	24.00	7.03	6.62	4.20	-7.35	34.50	0.99983
1990	25.63	6.52	7.45	5.18	-7.91	36.87	1.00347
1991	25.43	5.01	7.71	5.91	-7.83	36.22	1.01637
1992	28.07	6.41	7.94	6.74	-8.78	40.38	1.00720
1993	31.15	7.92	7.85	7.24	-10.06	44.10	1.00318
1994	34.88	10.41	8.03	8.49	-11.90	49.92	1.00205
1995	37.86	11.15	8.24	9.97	-13.30	53.92	1.00366
1996	41.26	13.11	8.64	11.32	-14.92	59.42	1.00406
1997	45.28	15.93	9.38	13.41	-17.52	66.48	1.00428
1998	50.53	18.96	10.03	14.07	-19.99	73.61	1.00401
1999	56.14	21.00	11.16	14.92	-22.48	80.75	1.00259
2000	61.39	23.01	12.05	16.76	-25.67	87.53	1.00212
2001	63.95	20.24	13.00	15.90	-25.09	88.00	1.00127

Notes : (a) consumption, (b) investment, (c) government, (d) export, (e) import

Figure 1. Real GDP growth rates since 1980 and its additive decomposition in percent-change



VI. Concluding remarks

The remarkable Reinsdorf-Balk transformation by Reinsdorf(1997), Reinsdorf, Diewert and Ehemann (2002) and Balk (2004) provides a linkage for the two important classes of index numbers such as the arithmetic mean index (or Lowe index) and the geometric mean index(or log-change index). In this paper we show that it provides an additive decomposition in percent-change of chain indexes. It has been thought impossible because the chain index is decomposable in additive way only for geometric mean indexes while log-changes have little practical meaning when time span is rather long by deviating from corresponding percent-change.

References

Balk, B.M., "Decomposition of Fisher Indexes," *Economics Letters*, 82, 2004 pp.107-113.
___________, "Curing the CPI's Substitution and New goods Bias, draft, July 1999
Barnett, W.A., Ki-Hong Choi, Sinclair, T.M. "Differential approach to index number theory" *Journal of Agricultural and Applied Economics* 35, (Henri Theil Memorial Issue), 2003, US. Southern Economic Association, pp.59-64
Diewert, W. Erwin, "The quadratic approximation lemma and decompositions of superlative indexes, *Journal of Economic and Social Measurement*, 28, 2002, pp.51-61
Diewert, W.E., Superlative Index Numbers and Consistency in Aggregation, *Econometrica*, 46, 1978 pp.883-900.
Diewert, W.E., Exact and Superlative Index Numbers, *Journal of Econometrics*, 4, 1976 pp.115-145.
Dumagan, J.C., "Comparing the superlative Törnqvist and Fisher ideal indexes," *Economics Letters*, 76, 2002 pp.251-258.
Ehemann, C., A. Katz, and B. Moulton, "The Chain Additivity Issue and the U.S. National Economic Accounts," *Journal of Economic and Social Measurement*, 28, 2002, pp.37-49
Hallerbach, W.G., "An alternative decomposition of the Fisher index", *Economics Letters*, 86, 2005, pp.147-152
Kohli, Ulrich, , "A Multiplicative Decomposition of the Fisher Index of Real GDP" in *Price and Productivity Measurement*, ed. E.W.Diewert, Bert Balk, Dennis Fixler, Kevin J. Fox and Alice Nakamura, 2007
Reinsdorf, Marshall. B. "Fixed Basket Versions of Geometric Mean Indexes and Measures of Consumer Surplus, FDIC, September, 1997
Reinsdorf, Marshall. B., W. Erwin Diewert, and Christian Ehemann, "Additive decompositions for Fisher, Törnqvist and geometric mean indexes, *Journal of Economic and Social Measurement*, 28, 2002, pp.51-61
Sato, K., "The Ideal log-change Index Number," *The Review of Economics and Statistics*, 58, 1976, pp.223-228.
Theil, H., "A New Index Number Formula," *The Review of Economics and Statistics*, 55, 1973 pp. 498-502.
_________ , *Introduction to Econometrics*, Prentice-Hall Inc. 1978
Thörnqvist, L., P.Vartia, and Y.O.Vartia, "How should relative changes be meaured?," *The American Statistician*, 39, 1985, pp.43-46
Vartia, Y.O., Ideal log-change Index Numbers, *Scandinavian Journal of Statistics*, 3, 1976, pp.121-126.
van IJzeren, J., "Over de plausibilteit van Fisher's ideal indices (On the plausibility of Fisher's ideal indices)" Statistische en Econometrische Onderzoekingen (C.B.S). Nieuwe Reeks 7, pp.104-115.