Systems of Index Numbers for International Price Comparisons Based on the Stochastic Approach

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Abstract

The main objective of the paper is to demonstrate that a number of widely used multilateral index numbers for international comparisons of purchasing power parities (PPPs) and real incomes can be derived using the stochastic approach. The paper introduces a new class of index numbers for international price comparisons and proves the existence and uniqueness of the new price index. The paper outlines a stochastic approach to generate the Ikle (1972), the Rao-weighted CPD (2005) and the new system of index numbers. The advantage of the stochastic approach is that we can derive standard errors for the estimates of the purchasing power parities (PPPs). The PPPs and the parameters of the stochastic model are estimated using a weighted maximum likelihood procedure. Estimates of PPPs and their standard errors for OECD countries using the proposed methods are presented.

The paper also outlines a method of moments approach to the estimation of PPPs under the stochastic approach. The paper shows how the Geary-Khamis system of multilateral index numbers can be derived using the stochastic approach thus providing a coherent framework for its derivation. Standard errors of the Geary-Khamis PPPs are presented in the paper.

JEL Classification: E31 and C19
Keywords: Purchasing Power Parities, International Prices, CPD, Gamma Distribution, Maximum Weighted Likelihood; Geary-Khamis method; Method of Moments

1. Introduction

There is considerable demand for reliable comparisons of real incomes between countries. In order to make incomes comparable across countries it is necessary to convert national income aggregates such as gross domestic product using appropriate currency converters. For many obvious reasons exchange rates are not considered appropriate and as such they do not reflect relative price levels in different countries.
Instead measures of spatial price levels in different countries, usually referred to as purchasing power parities (PPPs) of currencies, are employed. Much of the work on the compilation of PPP is principally under the auspices of the International Comparison Project/Program (ICP) undertaken jointly by a number of international organisations including the World Bank, United Nations, OECD and the European Union.

Purchasing power parities are computed using price data collected from the participating countries. PPP compilation within the ICP is undertaken at two levels, viz., at the basic heading level and at a more aggregated level. At the basic heading level price data are aggregated without any weights to yield PPPs for various basic headings. The basic heading PPPs are then aggregated to yield PPPs for higher level aggregates like consumption, investment and gross domestic product. The main focus of the paper is on the step involving the aggregation above the basic heading level where weights for each basic heading are available for all the countries.

A range of methods have been proposed in the literature by different authors to compute purchasing power parities for aggregation above the basic heading level. Some of the more popular ones are Geary-Khamis (Khamis 1970), Ikle (1972), Country-Product-Dummy (CPD) (Rao 1990, 2004, 2005; Diewert, 2005), Elteto-Koves-Szulc (EKS) (see e.g. Rao 2004). Balk (1996) has compared the analytical properties of more than 10 different methods for calculation of PPPs. Diewert (2005) has demonstrated that a number of commonly used formulae can be derived using the CPD method and Rao (2005) established that the Rao (1990) method for computing

PPPs is equivalent to the weighted CPD method. Thus a formal link between the stochastic approach to index numbers in the form of the CPD method and some of the more commonly used multilateral index number formulae has been established through the work of Diewert (2005) and Rao (2005). In the past there have been attempts to derive the Geary-Khamis method using stochastic approach (Rao and Selvanatha, 1999 and Diewert, 2005) but neither of these attempts have been successfully in providing a proper framework under the stochastic approach to derive the Geary-Khamis index and its standard errors. This problem is revisited and a solution is offered for the problem.

There are two principal objectives for the paper. The first objective of this paper is to further strengthen this link by showing that the multilateral price index number system introduced by Ikle (1972) can be derived from a stochastic modeling approach. In addition we consider a new variant of Ikle (1972) and Rao (1990) systems and show that it can also be easily incorporated into a stochastic model. These results are derived through the use of "weighted likelihood functions" which are necessary to consider stochastic specifications that involve distributions other than the normal or lognormal distributions implicit in the standard least squares approaches used along with the CPD model.

The second objective of the paper is to show that the Geary-Khamis multilateral system is indeed a method of moments estimator of the purchasing power parities and international prices within the same stochastic model that underpins the Rao, Ikle and other methods.

This paper is organized as follows: In Section 2 we introduce a new method for computing of purchasing power parities and we show its relationship to Rao (1990) and Ikle (1972) methods. We prove the existence and uniqueness of the new price index in Section 3. In Section 4 we introduce a stochastic model incorporating the new system and we provide a maximum likelihood approach to estimate the model. In Section 4 we do the same for Ikle index. The advantage of the stochastic approach is that we can derive standard errors for the estimates of the purchasing power parities, PPPs), this aspect is considered in Section 5. Section 6 presents estimates PPPs and their standard errors for OECD countries using the Rao, Ikle and the new methods of aggregation and the stochastic approach proposed here. Section 7 focuses on the derivation of the Geary-Khamis index as a method of moments estimator within the stochastic approach. The paper is concluded with a few remarks.

2- Notations and Definitions

Let p_{ij} and q_{ij} represent the price and the quantity of the jth commodity in the ith country respectively where $j = 1,...,M$ indexes the countries and $i = 1,...,N$ indexes the commodities. We assume that all the prices are strictly positive and all the quantities are non-negative with the minimum condition that for each i q_{ij} is strictly positive for at least one j; and for each j q_{ij} is strictly positive for at least one i. Also define PPP_j as purchasing power parity or the general price level in j-th country relative to a numeraire country and P_i as the world average price for the ith commodity. We also need the following systems of weights w_{ij} and w_{ij}^* in defining different systems of index numbers. These weights are defined as

$$w_{ij} = \frac{p_{ij} q_{ij}}{\sum_{i=1}^{N} p_{ij} q_{ij}} \quad \text{and} \quad w_{ij}^* = \frac{w_{ij}}{\sum_{j=1}^{M} w_{ij}} \qquad (1)$$

It is evident that $\sum_{i=1}^{N} w_{ij} = 1$ and $\sum_{j=1}^{M} w_{ij}^* = 1$.

With the above notations, Rao (1990) defines a system for international price comparisons as follows

$$PPP_j = \prod_{i=1}^{N} \left(\frac{p_{ij}}{P_i} \right)^{w_{ij}}$$

$$P_i = \prod_{j=1}^{M} \left(\frac{p_{ij}}{PPP_j} \right)^{w_{ij}^*} \qquad (2)$$

Following Balk (1996) another system proposed by Ikle (1972) can be written as:

$$\frac{1}{PPP_j} = \sum_{i=1}^{N} \left(\frac{p_i}{P_{ij}} w_{ij} \right)$$

$$\frac{1}{P_i} = \sum_{j=1}^{M} \left(\frac{PPP_j}{p_{ij}} w_{ij}^* \right) \tag{3}$$

Note that in Rao system, PPPs and world prices are defined as geometric means (Jevons type of price index) of some appropriate prices while in Ikle system harmonic means of the same prices have been used in a similar manner. Here, we propose a similar system of equations but using arithmetic means (Carli type of price index) as follows:

$$PPP_j = \sum_{i=1}^{N} \left(\frac{p_{ij}}{P_i} w_{ij} \right)$$

$$P_i = \sum_{j=1}^{M} \frac{p_{ij}}{PPP_j} w_{ij}^* \tag{4}$$

3- Existence and Uniqueness of the new Index

For both Rao and Ikle cases it has been shown that there are unique positive solutions for $\mathbf{P} = (P_1, P_2, ..., P_N)$ and $\mathbf{PPP} = (PPP_1, PPP_2, ..., PPP_M)$ in their systems (see Rao 1990 and Balk 1996). Following the same tradition we prove the existence and uniqueness of the new system. To do that, we use the following theorem from Morishima (1964, page 214).

Theorem (1): Let $G_i(x_1, x_2, ..., x_n)$ for $i = 1, ..., n$ satisfy following conditions:

(i) Homogeniety: functions $G_i(x_1, x_2, ..., x_n)$ for $i = 1, ..., n$ be homogenous of degree one;

(ii) Non-negativity: $G_i(.)s$ are defined for non-negative values of the arguments and are non-non-negative.

(iii) Monotonicity: for all $\mathbf{x} \le \mathbf{y}$, $G_i(\mathbf{x}) \le G_i(\mathbf{y})$

(iv) Indecomposibility: for any nonempty subset $\Omega \subset \{1,....,n\}$ the $x_i = y_i$ and $x_j \le y_j$ for $j \notin \Omega$ the exist at least one $i \in \Omega$ such that $G_i(x_1, x_2,...., x_n) \ne G_i(y_1, y_2,...., y_n)$.; and

There is a unique $\mathbf{x}^*$ (up to a positive scalar factor) and λ^* which solves the nonlinear eigen-value problem

$$G_i(x_1, x_2,....., x_n) = \lambda x_i \quad (i = 1,...., n)$$

Before presenting the main theorem concerning the existence and uniqueness of the proposed index i.e.

$$PPP_j = \sum_{i=1}^{N} \left(\frac{p_{ij}}{P_i} w_{ij} \right) \qquad (j = 1,....., M) \qquad (5.1)$$

$$P_i = \sum_{j=1}^{M} \frac{p_{ij}}{PPP_j} w_{ij}^* \qquad (i = 1,....., N-1) \qquad (5.2)$$

If we substitute P_i s from (5.2) in the first set of above equations (5.1) we have

$$PPP_j = \sum_{i=1}^{N} \frac{p_{ij}}{\sum_{j=1}^{M} \frac{p_{ij}}{PPP_j} w_{ij}^*} w_{ij} \quad (j = 1,....., M) \qquad (6)$$

Note that existence of a solution to (6) is equivalent to existence of a solution to the whole system (5.1) and (5.2). To prove that Let's define

$$G_j(PPP_1, PPP_2,....., PPP_M) = \sum_{i=1}^{N} \frac{p_{ij}}{\sum_{j=1}^{M} \frac{P_{ij}}{PPP_j} w_{ij}^*} w_{ij}$$

Theorem: (i) The system of equations (5) has a unique positive solution

As we showed above the system (5.1) and (5.2) can be reduced to

$$G_j(\mathbf{PPP}) = \sum_{i=1}^{N} \frac{p_{ij}}{\sum_{j=1}^{M} \frac{p_{ij}}{PPP_j} w_{ij}^*} w_{ij} \qquad (7)$$

It is easy to check that G_j satisfy all theconditions:

(i) G_j s is homogenous of degree one in $\mathbf{PPP}$

(ii) G_j s are defined over the non-negative values and are non-negative

(iii) G_j s are monotonic

(iv) The irreducibility is satisfied if there is at least one price for each commodity and at least one price for each country.

Therefore there is a unique $\mathbf{PPP}^*$ (up to a positive scalar factor) and λ^* which solves the following system of equations

$$\sum_{i=1}^{N} \frac{p_{ij}}{\sum_{j=1}^{M} \frac{p_{ij}}{PPP_j^*} w_{ij}^*} w_{ij} = \lambda^* PPP_j^*$$

The next step is to show that $\lambda^* = 1$. Not that we can write from

$$\lambda^* = \frac{1}{PPP_j^*} \sum_{i=1}^{N} \frac{p_{ij}}{\sum_{j=1}^{M} \frac{p_{ij}}{PPP_j^*} w_{ij}^*} w_{ij}$$

If we sums both sides of the above equations over $j = 1,\ldots,M$ we obtain

$$M\lambda^* = \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{\frac{p_{ij}}{PPP_j^*} w_{ij}}{\sum_{j=1}^{M} \frac{p_{ij}}{PPP_j^*} w_{ij}^*}$$

It is easy to see that

$$\sum_{i=1}^{N} \sum_{j=1}^{M} \frac{\frac{p_{ij}}{PPP_j^*} w_{ij}}{\sum_{j=1}^{M} \frac{p_{ij}}{PPP_j^*} w_{ij}^*} = \sum_{i=1}^{N} \sum_{j=1}^{M} w_{ij} = M$$

Therefore $\lambda^* = 1$ which proves the theorem. **Q.E.D**

4- Stochastic Approach to the Ikle Index and the New Index

To obtain the stochastic model incorporating the new index we follow Rao (2005) and Diewert (2005) to postulate that the observed price of j-th commodity in i-th country, p_{ij}, is the product of three components: the purchasing power parity (i.e. PPP_j); the price level of the j-th commodity relative to other commodities (i.e. P_i) and a random disturbance term u_{ij} as follows

$$p_{ij} = P_i PPP_j u_{ij} \qquad (8)$$

where $u_{ij}s$ are random disturbance terms which are independently and identically distributed. Rao (2005) has shown that Rao system (2) can be obtained as an estimator from the above model using a weighted least square argument after taking logs from both sides of the above equation. The same solution can be obtained by assuming a log-normal distribution for u_{ij} and using a maximum likelihood approach with weights attached to different observations.

In the following discussion, we explore alternative specifications for the distribution of u_{ij} which can be used in modeling the residuals of the CPD model in (8). In particular we use the gamma and inverted-gamma distributions and show that under these two specific distributions the resulting weighted maximum likelihood estimators coincide with the Ikle and the new system of index numbers.

Gamma distribution and the new Index

Here we assume u_{ij}s follows a gamma distribution as follows

$$u_{ij} \sim Gamma(r, r) \qquad (9)$$

where r is a parameter to be estimated. We combine (8) and (9) to write[2]

[2] One may notice the close association of the proposed model to what is known as a generalized linear model with gamma distribution. A generalized linear gamma regression may be defined as (see

$$\frac{p_{ij}}{P_i PPP_j} \sim Gamma(r,r) \qquad (10)$$

Our purpose here is to estimate parameters (i.e. P_i, PPP_j and r) using a maximum likelihood procedure. From the definition of the gamma density function we can easily show that

$$p_{ij} \sim \frac{r^r}{\Gamma(r)} \frac{{p_{ij}}^{r-1}}{P_i {PPP_j}^r} e^{-r\frac{p_{ij} r}{P_i PPP_j}} \qquad (11)$$

Therefore the log of density function can be written as

$$LnL_{ij} \propto r \ln r - \ln \Gamma(r) + (r-1) \ln p_{ij} - \ln P_i - r \ln PPP_j - r \frac{p_{ij} r}{P_i PPP_j} \qquad (12)$$

We can proceed with this (log-) density function and obtain estimates of the parameters of interest using the standard maximum likelihood procedure but we would like to incorporate the weights into the model as well. Use of weights is consistent with standard index number approach of weighting price relatives by their expenditure shares. This is also the approach used by Rao (2005) where weighted least squares method is employed.

One way of doing this is to use a weighted likelihood estimation procedure. Let's define the weighted likelihood function as

$$WL = \prod_{i=1}^{N} \prod_{j=1}^{M} L_{ij}^{w_{ij}/M} \qquad (13)$$

and therefore the weighted log-likelihood function becomes

McCullagh and Nelder 1989) $\frac{y_i}{\mathbf{x}_i \boldsymbol{\beta}} \sim Gamma(r,r)$. Our model is a nonlinear version of such a model.

$$LnWL = \sum_{i=1}^{N}\sum_{j=1}^{M}\frac{w_{ij}}{M}LnL_{ij} \qquad (14)$$

Then our weighted log- likelihood function becomes

$$\ln WL \propto (r-1)\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\ln p_{ij} - r\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\ln P_i - r\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\ln PPP_j - r\sum_{i=1}^{N}\sum_{j=1}^{M}\frac{p_{ij}w_{ij}}{P_i PPP_j} + r\ln r(\sum_{i=1}^{N}\sum_{j=1}^{M} w_{ij}) - \ln\Gamma(r)\sum_{i=1}^{N}\sum_{j=1}^{M} w_{ij} \qquad (15)$$

Note that the above function may not represent a density function therefore we don't interpret the estimation procedure as a maximum likelihood procedure. We rather interpret it as an M-estimation procedure (for more on M-Estimators and their properties see chapter 12 of Wooldridge 2002 or chapter 5 of Cameron and Trivedi 2005).

Maximization of this objective function is not particularly difficult. The only potential problem is the presence of a gamma function in the likelihood function however most of the existing software such as LIMDEP and GAUSS can handle maximization of the functions containing gamma functions fairly easily.

We can also derive the first order conditions from maximization of the above likelihood function as follows

$$-\frac{r\sum_{j=1}^{M} w_{ij}}{P_i} + \frac{r}{P_i^2}\sum_{j=1}^{M}\frac{p_{ij}w_{ij}}{PPP_j} = 0$$

$$-\frac{r\sum_{i=1}^{N} w_{ij}}{PPP_j} + \frac{r}{PPP_j^{\,2}}\sum_{i=1}^{N}\frac{p_{ij}w_{ij}}{P_i} = 0$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\ln p_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\ln P_i - \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}\ln PPP_j - \sum_{i=1}^{N}\sum_{j=1}^{M}\frac{p_{ij}w_{ij}}{P_i PPP_j} + M + M\ln r - M\frac{\partial}{\partial r}\ln\Gamma(r) = 0$$

From the above sets of equations we may obtain

$$P_i - \sum_{j=1}^{M} \frac{p_{ij} w_{ij}^*}{PPP_j} = 0$$

$$PPP_j - \sum_{i=1}^{N} \frac{p_{ij} w_{ij}}{P_i} = 0 \qquad (16)$$

$$\frac{\partial}{\partial r} \ln \Gamma(r) - \ln r = \frac{1}{M} (\sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \ln p_{ij} - \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \ln P_i - \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} \ln PPP_j - \sum_{i=1}^{N} \sum_{j=1}^{M} \frac{p_{ij} w_{ij}}{P_i PPP_j} + M)$$

As we can see the first two equations are the same as the system of equations we introduced as the new system and these equations do not depend upon the value of *r*.

Inverse Gamma Distribution and the Ikle Index

A similar procedure can be followed to obtain the stochastic model leading to Ikle index. In order to use the inverse-Gamma distribution, we rewrite the CPD model slightly differently. We use the reciprocal of the price and obtain:

$$\frac{1}{p_{ij}} = \frac{1}{P_i PPP_j} u_{ij} \qquad (17)$$

where u_{ij} s are random disturbance terms which are independently and identically and as before they are assumed to follow a gamma distribution

$$u_{ij} \sim Gamma(r, r) \qquad (18)$$

where r is a parameter to be estimated. Model in equation (17) differs from the model in equation (4) mainly in the specification of the disturbance term and how it enters the equation. One of the possible advantages of this model is that we do not have the inverse relationship between variance of p_{ij} and w_{ij}, We combine (17) and (18) to write

$$\frac{1}{p_{ij}} \propto \frac{r^r}{\Gamma(r)} \frac{(P_i PPP_j)^r}{p_{ij}^{\,r-1}} e^{-r \frac{P_i PPP_j}{p_{ij}}} \qquad (19)$$

Following the same procedure as we used in section (4) we may obtain the likelihood function as

$$lnL \propto -(r-1)\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} \ln p_{ij} + r\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} \ln P_i + r\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij} \ln PPP_j -$$

$$r\sum_{i=1}^{N}\sum_{j=1}^{M}\frac{P_i PPP_j w_{ij}}{p_{ij}} + r\ln r(\sum_{i=1}^{N}\sum_{j=1}^{M} w_{ij}) - \ln\Gamma(r)\sum_{i=1}^{N}\sum_{j=1}^{M} w_{ij} \qquad (20)$$

Taking derivative with respect to **PPP** and **P** yields the Ikle system of equations

$$\frac{1}{PPP_j} = \sum_{i=1}^{N}\left(\frac{P_i}{P_{ij}} w_{ij}\right)$$

$$\frac{1}{P_i} = \sum_{j=1}^{M}\frac{PPP_j}{P_{ij}} w_{ij}^{*} \qquad (21)$$

Thus the difference between Ikle and our newly proposed system in this paper is in the specification of the disturbance term.

5. Computation of Standard Errors

We have emphasized that the advantage of the stochastic approach to index numbers is to obtain standard errors for estimated indices. One might think that standard errors from conventional weighted least square or weighted maximum likelihood provided by standard software can be used for this purpose. But such standard errors may not be valid if cautions have not been made in deriving them.

To prove the point let's start with a general discussion of M- estimators and their variances. An M-Estimator $\hat{\boldsymbol{\theta}}$ is defined as an estimator that maximizes an objective function of the following form (See e.g. Cameron and Trivedi 2005)

$$Q_N(\boldsymbol{\theta}) = \frac{1}{N}\sum_{i=1}^{N} h_i(y_i, \mathbf{x}_i; \boldsymbol{\theta}) \qquad (22)$$

where y_i and $\mathbf{x}_i$ represent dependent and independent variables respectively. $\boldsymbol{\theta}$ is the vector of parameters to be estimated. It has been shown that $\hat{\boldsymbol{\theta}}$ has the following asymptotic distribution

$$\sqrt{N}(\hat{\boldsymbol{\theta}}-\boldsymbol{\theta}_0)\xrightarrow{d}\aleph[\mathbf{0},\mathbf{A}_0^{-1}\mathbf{B}_0\mathbf{A}_0^{-1}]$$

where

$$\mathbf{A}_0 = \text{plim}\frac{1}{N}\sum_{i=1}^{N}\frac{\partial^2 h_i}{\partial\boldsymbol{\theta}'\partial\boldsymbol{\theta}}\bigg|_{\boldsymbol{\theta}_0} \tag{23}$$

$$\mathbf{B}_0 = \text{plim}\ \frac{1}{N}\sum_{i=1}^{N}\ \frac{\partial h_i}{\partial\boldsymbol{\theta}}\bigg|_{\boldsymbol{\theta}_0}\ \frac{\partial h_i}{\partial\boldsymbol{\theta}'}\bigg|_{\boldsymbol{\theta}_0}$$

In practice, a consistent estimator can be obtained as

$$\mathbf{VAR}(\hat{\boldsymbol{\theta}}) = \frac{1}{N}\hat{\mathbf{A}}^{-1}\hat{\mathbf{B}}\hat{\mathbf{A}}^{-1} \tag{24}$$

where

$$\hat{\mathbf{A}} = \frac{1}{N}\sum_{i=1}^{N}\frac{\partial^2 h}{\partial\boldsymbol{\theta}'\partial\boldsymbol{\theta}}\bigg|_{\hat{\boldsymbol{\theta}}}$$

(25)

$$\hat{\mathbf{B}} = \frac{1}{N}\sum_{i=1}^{N}\frac{\partial h_i}{\partial\boldsymbol{\theta}}\bigg|_{\hat{\boldsymbol{\theta}}}\frac{\partial h_i}{\partial\boldsymbol{\theta}'}\bigg|_{\hat{\boldsymbol{\theta}}} \tag{26}$$

In some special cases like the maximum likelihood or nonlinear least square with homoscedastic errors it can be shown that $\mathbf{A}_0^{-1} = \mathbf{-B_0}$. In such cases the variance formula can be simplified to

$$\mathbf{VAR}(\hat{\boldsymbol{\theta}}) = -\frac{1}{N}\hat{\mathbf{A}}^{-1} \tag{27}$$

Many softwares use this formula as their default standard error formula. But in case of the problem studied in this paper this formula lead to incorrect standard errors for the estimated parameters and we must use the more general formula given by (23).

For example if we apply formula (27) to a the estimates from a weighted least squares regression we obtain following formula

$$\mathbf{VAR}(\hat{\boldsymbol{\theta}}) = \hat{\sigma}^2 (\mathbf{X}'\boldsymbol{\Omega}\mathbf{X})^{-1} \tag{28}$$

where $\boldsymbol{\Omega}$ is a diagonal matrix with weights on its diagonal which coincide the standard formula for weighted least square when there is heteroscedasticity in error term. However the correct formula for the variance estimator to be used in the case where we used weighted least squares when the disturbances are homoskedastic, is given by:

$$\mathbf{VAR}(\boldsymbol{\theta}) = \hat{\hat{\sigma}}^2 (\mathbf{X}'\boldsymbol{\Omega}\mathbf{X})^{-1} (\mathbf{X}'\boldsymbol{\Omega}'\boldsymbol{\Omega}\mathbf{X})(\mathbf{X}'\boldsymbol{\Omega}\mathbf{X})^{-1} \tag{29}$$

where $\hat{\hat{\sigma}}^2$ is obtained from the un-weighted regression. This formula is similar to that suggested in Rao (2004) for the computation of standard errors used the weighted CPD method.

6. Application to OECD data

In this section we present estimated PPPs and their standard errors derived using the three methods of aggregation discussed in the paper and the 1996 OECD data. The price information that we have is in the form of PPPs at the basic heading level for 158 basic headings, with US dollar used as the numeraire currency. In addition we have expenditure, in national currency units, for each basic heading in all the OECD countries. These nominal expenditures provide the expenditure share data used in deriving the weighted maximum likelihood estimators under alternative stochastic specification of the disturbances.

For weighted CPD estimates we have used the weighted least squares methodology as explained in Rao (2005). For Ikle and the new index we used the weighted maximum likelihood approach described in Section 4.

Table: MLE estimates of PPPs and SE's

Country	**MLE Estimates**					
	New Index		CPD		Ikle	
	PPP	S.E	PPP	S.E	PPP	S.E.

GER	1.887	0.136	2.034	0.144	2.187	0.147
FRA	6.092	0.429	6.554	0.455	7.035	0.466
ITA	1425.96	109.727	1504.02	115.509	1584.381	119.196
NLD	1.921	0.150	2.056	0.155	2.205	0.156
BEL	35.491	2.577	37.890	2.698	40.450	2.728
LUX	33.578	2.488	35.816	2.618	38.191	2.700
UK	0.603	0.043	0.642	0.044	0.682	0.045
IRE	0.637	0.051	0.669	0.055	0.696	0.060
DNK	8.525	0.586	9.131	0.615	9.762	0.631
GRC	180.470	13.452	188.482	13.891	196.640	14.005
SPA	112.414	8.304	118.546	8.606	124.799	8.738
PRT	126.043	10.400	129.037	10.994	130.317	12.002
AUT	12.770	0.881	13.730	0.928	14.728	0.948
SUI	2.050	0.168	2.183	0.177	2.320	0.180
SWE	9.424	0.686	10.075	0.720	10.758	0.742
FIN	6.159	0.432	6.598	0.453	7.070	0.462
ICE	86.828	7.000	89.541	6.975	92.329	6.810
NOR	8.807	0.684	9.238	0.736	9.642	0.764
TUR	6304.23	579.128	6321.42	544.907	6357.003	506.991
AUS	1.264	0.099	1.333	0.103	1.407	0.104
NZL	1.464	0.111	1.530	0.113	1.596	0.115
JAP	182.031	13.622	187.429	14.282	192.392	14.780
CAN	1.168	0.090	1.229	0.094	1.295	0.096
USA	1.0		1.0		1.0	

Results shown in the table clearly demonstrate the feasibility and comparability of the new approaches to the estimation of PPPs. As it can be seen, PPPs and their standard errors based on CPD, Ikle and the new index are all numerically close to each other. An additional phenomenon to note is that the PPPs based on the weighted CPD (or from the log-normal specification for the disturbances) appear to be bounded by PPP estimates from the new index and the Ikle index. However this seems to be only a coincidence. For example if we change the base country (for example to Australia) such a relation does not exist.

7. Derivation of Geary-Khamis (G-K) system using stochastic approach

The Geary-Khamis index due to Geary (1958) and Khamis (1970) and various other papers) is popular method of aggregation for international comparisons as it provides additively consistent international comparisons. The Geary-Khamis system is defined by the following system of interdependent equations:

$$\begin{cases} PPP_j = \dfrac{\sum_{i=1}^{n} p_{ij} q_{ij}}{\sum_{i=1}^{n} P_i q_{ij}} \\ P_i = \dfrac{\sum_{j=1}^{m} \left(p_{ij} q_{ij} \big/ PPP_j \right)}{\sum_{j=1}^{m} q_{ij}} \end{cases} \qquad (30)$$

The properties of the Geary-Khamis are widely discussed in the literature. Kravis et al, (1982) provide a comprehensive discussion of method and its salient features.

In the past there have been several attempts to cast the G-K method in a stochastic framework so that standard errors can be derived. One of the early attempts was due to Rao and Selvanathan (1992) but their approach is limited since the standard errors for PPPs were derived conditional on the knowledge of the international prices, P_i*'s*. Recently, Diewert (2005) attempted to derive the Geary-Khamis bilateral index using the stochastic approach based on the CPD method but the derivation is based on several ad hoc steps. In this paper, we show that the Geary-Khamis *PPP*'s are the method of moments estimators of the parameters of the CPD specification discussed in earlier sections of the paper. In particular, the approach used here recognises the non-additive nature of the CPD model and proposes the method of moments approach. These aspects are presented in the following subsections. In section 7.1 we discuss how a non-additive nonlinear system of equations can be estimated using a generalized method of moments. Section 7.2 applies this approach to the CPD model which is a non-additive model and shows how the arithmetic and the Geary-Khamis indices can be derived using this approach. A numerical illustration which presents the G-K *PPP*'s and their standard errors is included in Section 7.3.

7.1 Estimation of non-additive nonlinear models

Consider the following nonlinear regression model

$$r(y_i, \mathbf{x}_i, \boldsymbol{\beta}) = u_i \qquad (31)$$

where y_i represent the dependent variable, u_i represents the random errors, $r(y_i, \mathbf{x}_i, \boldsymbol{\beta})$ is a nonlinear function and $\mathbf{x_i}$ is a $1 \times L$ vector, $\boldsymbol{\beta}$ is a $K \times 1$ column vector, $i = 1,....,N$ indexes the number of observations and we also assume that $E(u_i) = 0$. We make a further assumption that the model is non-additive which means it can not be written as

$$y_i - g(\mathbf{x}_i, \boldsymbol{\beta}) = u_i$$

An additive model can be estimated using a nonlinear least square argument but it can be shown that a least square criterion does not provide consistent estimators for non-additive models (see e.g. Cameron and Trivedi 2005). How a non-additive model can be estimated?

An obvious starting point is to base the estimation of parameters in (31) on the moment conditions $E(\mathbf{X}'\mathbf{u}) = \mathbf{0}$ where $\mathbf{X}$ is the $N \times L$ matrix containing $\mathbf{x}_i$ s and $\mathbf{u}$ is an $N \times 1$ vector containing u_i s. However other moment conditions can be used. More generally we can base the estimation on the following K moment conditions:

$$E(\mathbf{R}(\mathbf{x},\boldsymbol{\beta})'\mathbf{u}) = \mathbf{0} \qquad (32)$$

where $\boldsymbol{R}$ is a $N \times K$ vector of functions of $\mathbf{X}$ and $\boldsymbol{\beta}$. By construction there are as many moment conditions as parameters therefore a method of moment estimator can be obtained by solving following sample moment conditions

$$\frac{1}{N}\mathbf{R}(\mathbf{X},\hat{\boldsymbol{\beta}})^{'}\mathbf{r}(\mathbf{y},\mathbf{X},\hat{\boldsymbol{\beta}}) = \mathbf{0} \qquad (33)$$

This estimator is asymptotically normal with variance matrix

$$Var(\hat{\boldsymbol{\beta}}_{MM}) = \hat{\sigma}^2\left[\hat{\mathbf{D}}'\hat{\mathbf{R}}\right]^{-1}\hat{\mathbf{R}}'\hat{\mathbf{R}}\left[\hat{\mathbf{R}}'\hat{\mathbf{D}}\right]^{-1} \qquad (34)$$

where $\hat{\mathbf{D}} = \left.\frac{\partial \mathbf{r}(\mathbf{y},\mathbf{X},\boldsymbol{\beta})}{\partial \boldsymbol{\beta}^{'}}\right|_{\hat{\boldsymbol{\beta}}}$, $\hat{\mathbf{R}} = \mathbf{R}(\mathbf{X},\hat{\boldsymbol{\beta}})$ and $\hat{\sigma}^2 = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{N}$

The main issue in the above estimation problem is the specification of $\mathbf{R}(\mathbf{X},\boldsymbol{\beta})$. It has been shown (see e.g. Davidson and Mackinnon 2004) that the most efficient choice is

$$\mathbf{R}(\mathbf{X},\boldsymbol{\beta})^{*} = E\left[\frac{\partial \mathbf{r}(\mathbf{y},\mathbf{X},\boldsymbol{\beta})'}{\partial \boldsymbol{\beta}} \mid \mathbf{X}\right] \qquad (35)$$

In general the expectation term in the right hand side can not be derived unless we make very strong distributional assumptions but fortunately for the type of models we consider in this paper it is tractable.

7.2 Estimation of PPPs and standard errors using GMM

To obtain PPP s and their standard errors based on an the stochastic model we follow Rao (2005) and Diewert (2005) to postulate that the observed price of j-th commodity in i-th country, p_{ij}, is the product of three components: the purchasing power parity (i.e. PPP_j); the price level of the j-th commodity relative to other commodities (i.e. P_i) and a random disturbance term u_{ij} as follows

$$p_{ij} = P_i PPP_j u_{ij}^{*} \qquad (36)$$

where $u_{ij}s$ are random disturbance terms which are independently and identically distributed. We also assume that $E(u_{ij}^{*}) = 1$. Model in equation (36) can be written in the following equivalent form

$$\frac{p_{ij}}{P_i PPP_j} - 1 = u_{ij} \qquad (37)$$

with $E(u_{ij}) = 0$. This is now in the form of a non-additive nonlinear regression model as introduced in the previous section and therefore we can use the estimation method in the previous section. Using the theory discussed in the previous section, the equations to be solved can be written as

$$\frac{1}{nm}\mathbf{R}'\mathbf{r} = \mathbf{0} \tag{38}$$

where $\mathbf{R}'$ is an $(n+m)\times(n\times m)$ matrix and according to (35) the most efficient choice for it can be defined as follows

$$\mathbf{R}' = \mathbf{E}\begin{bmatrix}
-\frac{p_{11}}{P_1^2PPP_1} & 0 & -\frac{p_{12}}{P_1^2PPP_2} & 0 & \cdots & -\frac{p_{1m}}{P_1^2PPP_m} & 0 \\
0 & -\frac{p_{n1}}{P_n^2PPP_1} & 0 & -\frac{p_{n2}}{P_n^2PPP_2} & \cdots & 0 & -\frac{p_{nm}}{P_n^2PPP_m} \\
 & & & \cdots\cdots & & & \\
-\frac{p_{11}}{P_1PPP_1^3} \;\; .. & -\frac{p_{n1}}{P_nPPP_1^3} & 0 & \cdots\cdots & & \cdots & 0 \\
 & \cdots & & \cdots & & & \\
0 & \cdots & \cdots & & 0 & -\frac{p_{1m}}{P_1PPP_m^3} \;\cdots & -\frac{p_{nm}}{P_nPPP_m^3}
\end{bmatrix}$$

and

$$r_{ij} = \frac{p_{ij}}{P_iPPP_j} - 1 \tag{39}$$

Considering the fact that

$$E\left[\frac{p_{ij}}{P_iPPP_j}\right] = 1 \tag{40}$$

We can write the equations in the following matrix form

$$\begin{bmatrix} -\frac{1}{P_1} & 0 & -\frac{1}{P_1} & 0 & \cdots & -\frac{1}{P_n} & 0 \\ & \ddots & & \ddots & & & \ddots \\ 0 & -\frac{1}{P_n} & 0 & -\frac{1}{P_n} & \cdots & 0 & -\frac{1}{P_n} \\ -\frac{1}{PPP_1} \ \cdots & -\frac{1}{PPP_1} & 0 & \cdots\cdots & & & 0 \\ & \cdots & & \cdots & & & \\ 0 & \cdots\cdots & \cdots\cdots & 0 & -\frac{1}{PPP_m} & \cdots & -\frac{1}{PPP_m} \end{bmatrix} \begin{bmatrix} \frac{p_{11}}{P_1\ PPP_1} - 1 \\ \frac{p_{12}}{P_2\ PPP_1} - 1 \\ \vdots \\ \frac{p_{nm}}{P_n\ PPP_m} - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

We can write the normal equations as follows

$$\begin{cases} -\frac{1}{P_i}\sum_{j=1}^{M}\left(\frac{p_{ij}}{P_i PPP_j} - 1\right) = 0 \\ -\frac{1}{PPP_j}\sum_{i=1}^{N}\left(\frac{p_{ij}}{P_i PPP_j} - 1\right) = 0 \end{cases} \Rightarrow \begin{cases} P_i = \frac{1}{m}\sum_{j=1}^{m}\left(\frac{p_{ij}}{P_i PPP_j}\right) \\ PPP_j = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{p_{ij}}{P_i PPP_j}\right) \end{cases} \qquad (41)$$

But this is exactly the arithmetic index introduced earlier in Section 2 of this paper.

According to the theory in the previous section the variance for the estimated price indexes can be obtained by

$$Var(\hat{\boldsymbol{\beta}}_{MM}) = \hat{\sigma}^2 \left[\hat{\mathbf{D}}'\hat{\mathbf{R}}\right]^{-1} \hat{\mathbf{R}}'\hat{\mathbf{R}} \left[\hat{\mathbf{R}}'\hat{\mathbf{D}}\right]^{-1} \qquad (42)$$

Where

$$
\mathbf{D}'=\begin{bmatrix}
-\frac{p_{11}}{P_1^2 PPP_1} & & \bigcirc & -\frac{p_{12}}{P_1^2 PPP_2} & & \bigcirc & \cdots & -\frac{p_{1m}}{P_1^2 PPP_m} & & \bigcirc \\
 & \ddots & & & \ddots & & & & \ddots & \\
\bigcirc & & -\frac{p_{n1}}{P_n^2 PPP_1} & \bigcirc & & -\frac{p_{n2}}{P_n^2 PPP_2} & \cdots & \bigcirc & & -\frac{p_{nm}}{P_n^2 PPP_m} \\
 & & & & \cdots & & & & & \\
-\frac{p_{11}}{P_1 PPP_1^2} & \cdots & -\frac{p_{n1}}{P_n PPP_1^2} & 0 & \cdots & & & \cdots & & 0 \\
 & & \vdots & & \vdots & & & & & \\
0 & \cdots & & \cdots & & & 0 & -\frac{p_{1m}}{P_1 PPP_{1m}^2} & \cdots & -\frac{p_{nm}}{P_n PPP_m^2}
\end{bmatrix}
$$

So far we haven’t introduced weights in our price index. One way doing this is to define the **R** matrix as follows

$$
\mathbf{R}'=\begin{bmatrix}
-\frac{w_{11}}{P_1} & & \bigcirc & -\frac{w_{12}}{P_1} & & \bigcirc & \cdots & -\frac{w_{1m}}{P_n} & & \bigcirc \\
 & \ddots & & & \ddots & & & & \ddots & \\
\bigcirc & & -\frac{w_{n1}}{P_n} & \bigcirc & & -\frac{w_{n2}}{P_n} & \cdots & \bigcirc & & -\frac{w_{nm}}{P_n} \\
-\frac{w_{11}}{PPP_1} & \cdots & -\frac{w_{n1}}{PPP_1} & 0 & \cdots & & & & & 0 \\
 & & \vdots & & \vdots & & & & & \\
0 & \cdots & & \cdots & & & 0 & -\frac{w_{1m}}{PPP_m} & \cdots & -\frac{w_{nm}}{PPP_m}
\end{bmatrix}
$$

This definition for **R** matrix results in the following system of equations which coincides the weighted version of the arithmetic index

$$
\begin{cases}
PPP_j = \sum_{i=1}^{N}\left(\frac{p_{ij}}{P_i} w_{ij}\right) \\
P_i = \sum_{j=1}^{M} \frac{p_{ij}}{PPP_j} w_{ij}^{*}
\end{cases}
\tag{43}
$$

This might be criticized on the ground that the weights might be correlated with the disturbance term but this is standard in price index analysis; Diewert (2005) and Rao (2004) have used weighted least squares in similar manner.

7.3 Derivation of the Geary-Khamis PPPs and standard errors[i]

Consider again estimation of the following model

$$\frac{p_{ij}}{P_i PPP_j} - 1 = u_{ij}$$

As we said in the previous sections we can based our estimation based on following moment conditions

$$E\left[\mathbf{R}^{'}\mathbf{u}\right] = \mathbf{0}$$

and accordingly following sample moment conditions

$$\frac{1}{nm}\mathbf{R'r} = \mathbf{0}$$

Different definitions for $\mathbf{R}$ can lead to different estimator. As long as **R** is not correlated with **u** the estimator is consistent. We make a slight modification in the definition of **R** in the previous section as follows

$$\mathbf{R'} = \begin{bmatrix} -\frac{1}{P_1} & & 0 & -\frac{1}{P_1} & & 0 & \cdots & -\frac{1}{P_n} & & 0 \\ & 0 & -\frac{1}{P_n} & & 0 & -\frac{1}{P_n} & \cdots & & 0 & -\frac{1}{P_n} \\ -\frac{P_1}{PPP_1} & \cdots & -\frac{P_n}{PPP_1} & 0 & & \cdots & & & & 0 \\ & & \cdots & & & \cdots & & & & \\ 0 & \cdots & & \cdots & & 0 & & -\frac{P_1}{PPP_m} & \cdots & -\frac{P_n}{PPP_m} \end{bmatrix}$$

It is east to see that **R** is not correlated with **u** because **P** and **PPP** are parameters of the model to be estimated. (Note also that P_i s are close to one and therefore this matrix does not differ very much from the one in the last section). This definition for **R** results in the following equations

$$\begin{cases} \sum_{i=1}^{n} P_i \left(\frac{p_{ij}}{P_i PPP_j} - 1 \right) = 0 \\ \sum_{j=1}^{N} 1 \left(\frac{p_{ij}}{P_i PPP_j} - 1 \right) = 0 \end{cases} \Rightarrow \begin{cases} PPP_j = \dfrac{\sum_{i=1}^{n} p_{ij}}{\sum_{i=1}^{n} P_i} \\ P_i = \dfrac{1}{m} \sum_{j=1}^{m} \left(\dfrac{p_{ij}}{PPP_j} \right) \end{cases}$$

But this is the un-weighted Geary-Khamis price index. We can derive the weighted price index by defining

$$\mathbf{R'} = \begin{bmatrix} -\frac{q_{11}}{P_1} & 0 & -\frac{q_{12}}{P_1} & 0 & \cdots & -\frac{q_{1m}}{P_n} & 0 \\ 0 & -\frac{q_{n1}}{P_n} & 0 & -\frac{q_{n2}}{P_n} & \cdots & 0 & -\frac{q_{nm}}{P_n} \\ -\frac{q_{11}P_1}{PPP_1} & \cdots & -\frac{q_{n1}P_n}{PPP_1} & 0 & \cdots & & 0 \\ & & \cdots & & \cdots & & \\ 0 & \cdots & \cdots & 0 & -\frac{q_{1m}P_1}{PPP_m} & \cdots & -\frac{q_{nm}P_n}{PPP_m} \end{bmatrix}$$

This results in the following system of equations

$$\begin{cases} PPP_j = \dfrac{\sum_{i=1}^{n} p_{ij} q_{ij}}{\sum_{i=1}^{n} P_i q_{ij}} \\ P_i = \dfrac{\sum_{j=1}^{m} \left(p_{ij} q_{ij} / PPP_j \right)}{\sum_{j=1}^{m} q_{ij}} \end{cases}$$

which are identical to the equations that define the Geary-Khamis system. Thus it is clear that the G-K PPPs and P_i*'s* are the method of moments (weighted) estimators of the parameters of the CPD model.

As usual the standard errors for the estimated indexes can be obtained using following formula

$$Var(\hat{\boldsymbol{\beta}}_{MM}) = \hat{\sigma}^2 \left[\hat{\mathbf{D}}'\hat{\mathbf{R}}\right]^{-1} \hat{\mathbf{R}}'\hat{\mathbf{R}} \left[\hat{\mathbf{R}}'\hat{\mathbf{D}}\right]^{-1}$$

where $\mathbf{D}_{ij}$ s are the same as in the previous section.

*7.4 Empirical Illustration using OECD data*i

In this section we use the same 1996 OECD data described in Section 6 to demonstrate the feasibility of the method of moments approach described in this section and also present the estimates and standard errors of the G-K PPPs.

Table 2 shows the estimated PPPs and their standard errors based on: (i) arithmetic index using GMM; and (ii) Geary -Khamis using the method introduced in this paper. The standard errors of the arithmetic index based on the MLE approach discussed in Sections 4 and 5 of this paper are also presented.

Table1: Estimates of PPPs and SE's

	Arithmetic Index	GMM SE Arithmetic	MLE SE Arithematic	G-K Index	GMM SE G-K
GER	1.878	0.109442	0.136	2.08316	0.15474
FRA	6.067	0.606755	0.429	6.679491	0.516194
ITA	1419	79.25337	109.727	1537.168	129.5046
NLD	1.909	0.11156	0.150	2.032161	0.156602
BEL	35.3	1.946125	2.577	38.70436	2.700867
LUX	33.35	2.454269	2.488	36.7877	3.446165
UK	0.5996	0.036311	0.043	0.679564	0.053761

IRE	0.633	0.037709	0.051	0.657754	0.056569
DNK	8.481	0.591807	0.586	9.457703	0.872669
GRC	179.5	9.271153	13.452	187.3352	13.14857
SPA	111.8	7.726502	8.304	122.1712	10.59001
PRT	125.4	6.56711	10.400	124.7745	9.307088
AUT	12.71	0.731266	0.881	14.40264	1.098328
SUI	2.037	0.146331	0.168	2.220059	0.179608
SWE	9.382	0.726701	0.686	10.56069	1.024583
FIN	6.12	0.404593	0.432	6.895726	0.638499
ICE	86.15	6.142211	7.000	90.02853	9.473389
NOR	8.751	0.457666	0.684	9.119335	0.764748
TUR	6251	393.9744	579.128	5967.556	549.1221
AUS	1.259	0.08598	0.099	1.351173	0.106996
NZL	1.455	0.106893	0.111	1.545069	0.140098
JAP	181	12.52263	13.622	179.0048	15.83708
CAN	1.16	0.085695	0.090	1.271441	0.115112
USA	1.0			1	

The results from the table are consistent with the expectations. The standard errors for the arithmetic index using GMM is slightly more efficient than MLE. This could be because GMM is robust to the choice of distribution for the error term and the standard errors for the Geary-Khamis using the method proposed here are higher than the other two which is expected because it is not the most efficient estimator based on our stochastic specification.

8. Concluding remarks

The paper has proposed a straightforward extension to two known multilateral methods due to Ikle (1972) and Rao (1990). The new index uses weighted arithmetic averages to define PPPs and international prices, P_i's, instead of harmonic and geometric averages used respectively in Ikle and Rao specifications. The paper has also established that all the three indexes can be shown to be the weighted maximum likelihood estimators of the CPD model when the disturbances follow lognormal, gamma or the inverse gamma distributions. Derivation of the indices using the stochastic approach makes it possible to derive appropriate standard errors for the Ikle and the new index proposed here. Further, given that all these indexes are generated by the same CPD model but with alternative disturbance specifications it allows us to test for the distributional assumptions underlying these three methods and use such specification tests to choose between alternative methods. Further work is necessary to see if it is possible to explore other specifications for the distribution of the disturbance and the index number formulae resulting from such specifications. The paper also outlines the approach necessary to compute the true standard errors of PPPs when weighted maximum likelihood methods are used.

The paper has also shown that the commonly used Geary-Khamis PPPs can be derived from the CPD model and the stochastic approach described here. In particular, the G-K PPPs are shown to be weighted method of moments (MOM) estimators of the parameters of the CPD model.

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